Unit - III

Two Random Variables

__Two random variables__: Let, S be sample space associated with random experiment E and random variables X & Y be two functions assigning real no. to each outcome of the sample space, then X, Y are called two-dimensional random variable

Two types of Dimensional random variables:

1) Discrete
2) Continuous.

__Bi-variate Distributions__:

- __PRP mass function of (X,Y)__: If (X,Y) is a two-dimensional discrete random variable such that $P(X=x_i, Y=y_i)=P_{ij}$ then P_{ij} is called prp mass function of (X,Y) provided following conditions are satisfied.

(i) $P_{ij} \geq 0$

(ii) $\sum_i \sum_j P_{ij} = 1$

__Joint prp density function__: If (X,Y) is a 2D-continuous R.V such that $P(x-\frac{dx}{2} \leq X \leq x+\frac{dx}{2}, y-\frac{dy}{2} \leq y \leq y+\frac{dy}{2})$. then $f(x,y)$ is called joint pdf of (x,y) provided following conditions are satisfied

(i) $f(x,y) \geq 0$

(ii) $\iint f(x,y)\,dx\,dy = 1$

__Commulative Distribution function__ (CDF): If (X,Y) are 2D-random variable either discrete or continuous then $F(x,y) = P[X \leq x, Y \leq y]$ is called CDF of (x,y)

RV

__Marginal prp distribution of (X,Y)__: If (X,Y) be any 2D-R.V (discrete) then marginal distribution function is given by $P(X=x_i) = \sum_{j=1}^{n} P_{ij}$.

Marginal distribution function of X is given as

$$P(Y=y_j)=\sum_{i=1} p_{ij}$$

<u>Marginal density function</u> of <u>(X,Y)</u>: If (X,Y) is a 2D contin R.V then marginal function of X is given by

$$f_X(x)=\int f(x,y)\,dy$$

$$f_Y(y)=\int f(x,y)\,dx$$

<u>Independent Random Variable</u>: If (X,Y) is a 2D discrete R.V such that $P_{i\cdot} \cdot P_{\cdot j} = P_{ij}$ then X and Y are said to be independent R.V

- If (X,Y) is a 2D continuous R.V such that $f_X(x)\cdot f_Y(y)=f(x,y)$ then x, y are said to be independent R.V.

<u>Conditional prp distribution</u>: If (X,Y) is a 2D discrete R.V then conditional prp is given as

$$P(X=x_i/Y=y_j)=\frac{P[(X=x_i)\cap(Y=y_j)]}{P(Y=y_j)}$$

CPD of $Y=y_j$ is given as $X=x_i$

$$P(Y=y_j/X=x_i)=\frac{P[(X=x_i)\cap(Y=y_j)]}{P(X=x_i)}$$

<u>Conditional density function</u>: If (X,Y) is a 2D continuous R.V, then CDF of X is given by

$$f(X/y)=\frac{f(x,y)}{f_Y(y)}$$

C.D.F of Y is given by $f(Y/x)=\dfrac{f(x,y)}{f_X(y)}$

Q) For the adjoining bivalent prp discrete func of (X,Y)

7-8M

X\Y	1	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32
1	1/16	1/16	1/8	1/8	1/8	1/8
2	1/32	1/32	1/64	1/64	0	2/64

Find (i) $P(X\le 1, Y=2)$

(ii) $P(X\le 1)$

(iii) $P(Y\le 3)$

(iv) $P(X<3, Y\le 4)$

(v) $P(X\le 1 / Y\le 3)$

(vi) $P(Y\le 3/X\le 1)$

(vii) $P(X+Y\le 4)$

sol (i) $P(X \le 1, Y=2) = \sum_{x_i=0}^{1} P(X=x_i, Y=2)$

$= P(X=0, Y=2) + P(X=1, Y=2)$

$= 0 + \frac{1}{16}$

$= \frac{1}{16}$

(ii) $P(X \le 1) = \sum_{x_i=0}^{1} \sum_{y_i=1}^{6} P(X=x_i, Y=y_i)$

$= \sum_{y_i=1}^{6} P(X=0; Y=y_i) + \sum_{y_i=1}^{6} P(X=1, Y=y_i)$

$= \left(0+0+\frac{1}{32}+\frac{2}{32}+\frac{2}{32}+\frac{3}{32}\right) + \left(\frac{1}{16}+\frac{1}{16}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)$

$= \frac{8}{32} + \frac{10}{16}$

$= \frac{28}{32}$

$= \frac{7}{8}$

(iii) $P(Y \le 3) = \sum_{x_i=0}^{2} \sum_{y_i=1}^{3} P(X=x_i, Y=y_i)$

$= \sum_{y_j=1}^{3} P(X=0, Y=y_j) + \sum_{y_j=1}^{3} P(X=1, Y=y_j) + \sum_{y_j=1}^{3} P(X=2, Y=y_j)$

$= \left(0+0+\frac{1}{32}\right) + \left(\frac{1}{16}+\frac{1}{16}+\frac{1}{8}\right) + \left(\frac{1}{32}+\frac{1}{32}+\frac{1}{64}\right)$

$= \frac{1}{32} + \frac{4}{16} + \frac{5}{64} = \frac{2+16+5}{64}$

$= \frac{23}{64}$

(iv) $P(X<3, Y \le 4) = \sum_{x_i=0}^{2} \sum_{y_j=1}^{4} P(X=x_i, Y=y_j)$

$= \left(0+0+\frac{1}{32}+\frac{2}{32}\right) + \left(\frac{1}{16}+\frac{1}{16}+\frac{1}{8}+\frac{1}{8}\right)$

$+ \frac{1}{32}+\frac{1}{32}+\frac{1}{64}+\frac{1}{64}$

$= \frac{9}{16}$

(v) $P(X \le 1 / Y \le 3) = \frac{P(X \le 1, Y \le 3)}{P(Y \le 3)}$

$= \frac{\sum_{x_i=0}^{1} \sum_{y_j=1}^{3} P(X=x_i, Y=y_j)}{\sum_{x_i=0}^{2} \sum_{y_j=1}^{3} P(X=x_i, Y=y_j)}$

$= \frac{\left(0+0+\frac{1}{32}\right)+\left(\frac{1}{16}+\frac{1}{16}+\frac{1}{8}\right)}{23/64}$

$= \frac{7/32}{23/64} = \frac{18}{23}$

(vi) $P(Y \le 3 / X \le 1) = \dfrac{P(X \le 1, Y \le 3)}{P(X \le 1)}$

$= \dfrac{9/32}{7/8} = \dfrac{9}{32} \times \dfrac{8}{7}$

$= \dfrac{9}{28}$

(vii) $P(X+Y \le 4) = \sum_{y_j=1}^{4} P(X=0, Y=y_i) + \sum_{y_i=1}^{3} P(X=1, Y=y_j) + \sum_{y_j=1}^{2} P(X=2; Y=y_i)$

$= \left(0+0+\frac{1}{32}+\frac{2}{32}\right) + \left(\frac{1}{16}+\frac{1}{16}+\frac{1}{8}\right) + \left(\frac{1}{32}+\frac{1}{32}\right)$

$= \dfrac{13}{32}$ x, y 0,0 1,1 2,2, 3,3 →

- 8) PRP distribution function is given of as
$P(X, Y) = k(2x + 3y)$; $X = 0, 1, 2$; $Y = 1, 2, 3$
(i) Find k $k(2(0)+3(1))$
(ii) Marginal distributive func of X and Y.

x/y	1	2	3
0	3k	6k	9k
1	5k	8k	11k
2	7k	10k	13k

Sol: (i) k

$\sum_{x_i=0}^{2} \sum_{y_j=1}^{3} P(X=x_i, Y=y_i) \Rightarrow k[3] + k[6] + k[9] + k[5] + k[8] + k[11] + k[7] + k[10] + k[13]$

$\Rightarrow k[72) = 1$

$\Rightarrow k = \dfrac{1}{72}$ $[\because \sum P(X=x_i, Y=y_j) = 1]$

X/Y	1	2	3	$P_X(X)$
0	3k	6k	9k	18k
1	5k	8k	11k	24k
2	7k	10k	13k	30k

(ii) $P(X=0) \sum_{j=1}^{3} P_{ij} = 3k + 6k + 9k = 18k = \dfrac{18}{72} = \dfrac{1}{4}$

$P(X=1) \sum_{j=1}^{3} P_{ij} = 5k + 8k + 11k = 24k = \dfrac{24}{72} = \dfrac{1}{3}$

$P(X=2) \sum_{j=1}^{3} P_{ij} = 7k + 10k + 13k = \dfrac{5}{12}$

Q) Joint PDF of (X,Y) is $f(x,y) = \begin{cases} k e^{-(x+y)} ; & x \geq 0, y \geq 0 \\ 0 ; & o/w \end{cases}$ 3, 83

(i) Find k

(ii) Marginal density function of X, Y

(iii) Show X, Y are independent R.V

(iv) $P(0 \leq X \leq 2, 2 \leq y \leq 3)$

$\int e$

Sol. (i) $\int_{x=0}^{\infty} \int_{y=0}^{\infty} k \cdot e^{-(x+y)} dy\, dx = 1$

$\Rightarrow k \int_{x=0}^{\infty} \boxed{\int_{y=0}^{\infty} e^{-x} \cdot e^{-y} dy} dx = 1$

$\Rightarrow k \int_{x=0}^{\infty} e^{-x} \left(\frac{e^{-y}}{-1}\right)_0^{\infty} dx = 1$

$k[0+1] \int_{x=0}^{\infty} e^{-x} dx = 1$

$k\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1$

$k[0+1] = 1$

$k = 1$

NOTE:

$(UL - LL)$

$\left(\frac{e^{-\infty}}{-1}\right) - \left(\frac{e^{-0}}{-1}\right)$

$0 + \frac{1}{e^0}$

$0 + 1$

$e^0 = 1$

$e^{\infty} = \infty$

$e^{-\infty} = 0$

$\frac{1}{e^{\infty}} = \frac{1}{\infty}$

(ii) Marginal density func of X

$f_X(x) = \int f(x,y) dy$

$= \int_{y=0}^{\infty} e^{-x} e^{-y} dy$

$= e^{-x} \left[\frac{e^{-y}}{-1}\right]_0^{\infty}$

$= e^{-x}(0+1)$

$= e^{-x}$

$e^{-x} \cdot e^{-y} = e^{-(x+y)}$

Marginal density func. of Y

$f_Y(y) = \int_0^{\infty} f(x,y)\, dx$

$= \int_0^{\infty} e^{-(x+y)} dx$

$= e^{-y} \left[\frac{e^{-x}}{-1}\right]_0^{\infty}$

$= e^{-y}(0+1)$

$= e^{-y}$

$\int_0^{\infty} e^{-y} dy = \left[\frac{e^{-y}}{-1}\right]_0^{\infty}$

$= \frac{e^{-\infty}}{-1} - \frac{e^{-0}}{-1}$

$= 0 + 1 = 1$

(iii) X and Y are independent if $f_X(x) \times f_Y(y) = f(x, y)$

84

$$f_X(x) f_Y(y) = e^{-x} e^{-y}$$
$$= e^{-(x+y)}$$
$$= f(x+y).$$

$\therefore$ X and Y are independent R.V

$I_{dy} = \int f(x,y)\, dx\, dy$

$= \int f(x) dx \times \int f(y)$

(iv) $P(0 \le X \le 2 ; 2 \le y \le 3)$

Sol $\int_{x=0}^{2} \int_{y=2}^{3} e^{-(x+y)} dy\, dx = \int_{x=0}^{2} e^{-x} \int_{y=2}^{3} e^{-y} dy\, dx$

$$= \int_{x=0}^{2} e^{-x} \left(\frac{e^{-y}}{-1}\right)_2^3 dx.$$
$$= \int_{x=0}^{2} e^{-x}(-e^{-3} + e^{-2}) dx$$
$$= \int_{x=0}^{2} (e^{-2} - e^{-3}) e^{-x} dx.$$
$$= e^{-2} - e^{-3} \left[\frac{e^{-x}}{-1}\right]_0^2$$
$$= (e^{-2} - e^{-3})[-e^{-2} + 1]$$
$$= (e^{-2} - e^{-3})(1 - e^{-2})$$
$$= 0.07397$$

Q) Joint PDF of (X, Y) is $f(x,y) = \begin{cases} a(2x + y^2) ; & 0 \le x \le 2 \\ & ; 2 \le y \le 4 \\ 0 & ; \text{otherwise} \end{cases}$

(i) Find a

(ii) $P(X \le 1, Y \le 3)$

$P(X \le 1, Y < 3)$

Sol $\int_{x=0}^{2} \int_{y=2}^{4} a(2x + y^2) dy\, dx = 1$

$$\Rightarrow a \int_{x=0}^{2} \int_{y=2}^{4} (2x + y^2) dy\, dx = 1$$

~~$\Rightarrow a \int_{a=0}^{2} \int_{y=2}^{4} (2x + v$~~ $\Rightarrow a \int_{x=0}^{2} 2x(4-2) + \left(\frac{y^3}{3}\right)_2^4 dx = 1$

$$\Rightarrow a \int_{x=0}^{2} 2x(4-2) + \left(\frac{y^3}{3}\right)_2^4 dx = 1$$
$$\Rightarrow a \int_{x=0}^{2} 4x + \frac{56}{3}\, dx = 1$$
$$a\left[4\left(\frac{x^2}{2}\right)_0^2 + \frac{56}{3}(2)\right] = 1$$
$$a\left[8 + \frac{112}{3}\right] = 1$$
$$a = \frac{3}{136}$$

(ii) $P(X \leq 1, y \leq 3)$

Sol $\int_{x=0}^{1}\int_{y=2}^{3} \frac{3}{136}(2x+y^2)\,dy\,dx = \frac{3}{136}\int_{x=0}^{1}\int_{y=2}^{3}(2x+y^3)\,dy\,dx$

$$= \frac{3}{136}\int_{x=0}^{1} 2x(1) + \left(\frac{y^3}{3}\right)_2^3 dx.$$

85

$$= \frac{3}{136}\int_{x=0}^{1} 2x + \left(9 - \frac{8}{3}\right)dx.$$

$$= \frac{3}{136}\int_{x=0}^{1}\left(2x + \frac{11}{3}\right)dx.$$

$$= \frac{3}{136}\left[2\left(\frac{x^2}{2}\right)_0^1 + \frac{19}{3}\right]$$

$$= \frac{3}{136}\left[2\left(\frac{1}{2}\right) + \frac{19}{3}\right]$$

$$= \frac{11}{68}.$$

<u>Two functions</u> <u>of two</u> <u>R.V</u>: If (X,Y) is a 2D R.V with joint pdf $f_{xy}(x,y)$ and if $U=g(x,y)$ and $V=h(x,y)$ are other two R.V then the joint pdf of (U,V) is given by $f_{UV}(x,y) = |J|\, f_{xy}(x,y)$ where $J = \frac{\partial(x,y)}{\partial(u,v)}$ is the Jacobian of the transformation and is given by $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

Q) If the joint pdf of (x,y) is given as

$$f_{xy}(x,y) = \begin{cases} x+y \,; & 0 \leq x \leq 1 \\ & 0 \leq y \leq 1 \\ 0 \,; & \text{otherwise} \end{cases}$$

Find the pdf of $U = XY$.

Sol $f_{UV}(U,V) = |J|\, f_{xy}(x,y)$

Introducing auxillary R.V $V=y$

$$U = xy$$

$$U = xv \quad [\because y = v]$$

$$x = \frac{u}{v} \qquad y = v$$

$$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

$\frac{\partial x}{\partial u} = \frac{1}{v}$ $\quad \frac{\partial x}{\partial v} = -\frac{u}{v^2}$; $\frac{\partial y}{\partial u} = 0$; $\frac{\partial y}{\partial v} = 1$

$$J = \begin{vmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{vmatrix}$$

$$J = \frac{1}{V} + \frac{u}{V^2}(0)$$

86 $$J = \frac{1}{V}$$

$$f_{uv}(u,v) = |J|\, f_{xy}(x,y)$$
$$= \frac{1}{V}(x+y)$$
$$= \frac{1}{V}\left(\frac{u}{V} + v\right)$$
$$= \frac{u+V^2}{V^2}$$
$$= \frac{u}{V^2} + 1$$

$0 \le x \le 1 \;;\; 0 \le y \le 1$

$0 \le \frac{u}{V} \le 1 \;;\; 0 \le V \le 1$

$0 \le u \le V$

$$\therefore f_{uv}(u,v) = \begin{cases} \frac{u}{V} + 1 \;; & 0 \le u \le V \\ & 0 \le u \le 1 \\ 0 \;; & o/w \end{cases}$$

$$f_u(u) = \int_V f_{uv}(u,v)\, dv.$$

$0 \le u \le v$

$0 \le v \le 1 \Rightarrow u \le v \le 1 \rightarrow$ Range space of v.

$$f_u(x) = \int_u^1 \left(\frac{u}{v^2} + 1\right) dv.$$
$$= \int_u^1 \left(\frac{u}{v^2} + 1\right) dv$$
$$= \int_u^1 (uv^{-2} + 1)\, dv.$$
$$= \left[\frac{uv^{-1}}{-1} + v\right]_u^1$$

$$f_u(u) = [-(u-1) + 1 - u]$$
$$= [-u + 1 + 1 - u]$$
$$= 2 - 2u$$
$$= 2(1-u)$$

Q) If x & y each follow are exponential distribution with parameter 1 and are independent. Find pdf $(u = x - y)$

sol: $f_x(x) = \lambda e^{-\lambda x}, \; x \ge 0$

$\lambda = 1$

$f_x(x) = e^{-x} \;;\; x \ge 0$

$f_Y(y) = \lambda e^{-\lambda y}$; $y \geq 0$

$\lambda = 1$

$f_Y(y) = e^{-y}$; $y \geq 0$ 87

$f(x,y) = f_X(x) \cdot f_Y(y)$ $\quad \because$ X and Y are independent

$f_{XY}(x,y) = e^{-x} \cdot e^{-y}$; $x \geq 0$
$y \geq 0$

0 ; otherwise

$f_{UV}(u,v) = |J| \, f_{XY}(x,y)$

Introducing auxillary R.V $v = y$

$u = x - y$

$x = u + y \qquad y = v$

$J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} \qquad \frac{\partial u}{\partial x} = 1 \quad \frac{\partial y}{\partial u} = 0 \qquad \frac{\partial x}{\partial v} = 1 \quad \frac{\partial u}{\partial y} = 1$

$J = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \qquad J = 1$

$x = u + y$

$u = x - y \quad y = v$

$v = y$

$\frac{\partial u}{\partial x} = 1 \quad \frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = -1 \quad \frac{\partial v}{\partial y} = 1$

$f_{UV}(u,v) = |J| \, f_{XY}(x,y)$.

$= e^{-(x+y)}$

$= e^{-(u+y+v)}$

$= e^{-(u+2v)}$

$x \geq 0 \qquad y \geq 0$

$x + u \geq 0 \qquad v \geq 0 \qquad u = x - y$

$u \geq -v \qquad v \geq 0 \qquad x = u + y$

$f_U(u) = \int_{-u}^{\infty} f_{UV}(u,v)\, dv \qquad = u + v \geq 0 \quad u \geq -v$

$v \geq -u$ $\qquad u \geq -v \Rightarrow v \leq -u$

$f_U(u) = \int_{-u}^{\infty} f_{UV}(x,v)\, dv$

$u + v \geq 0$

$v \geq -u$

$f_X(u) = \int_{-u}^{\infty} e^{-(u+2v)} dv$.

$f_U(u) = e^{-u} \int_{-u}^{\infty} e^{-2v} dv$.

$= \frac{-e^{-u}}{2} \left[e^{-2v} \right]_{-u}^{\infty}$

$= \frac{-e^{-u}}{2} \left[0 - e^{2u} \right]$

$= \frac{e^{-u} \, e^{2u}}{2} = \frac{e^{u}}{2}$ — (1) $\qquad \left[\frac{e^{-4+2}}{2} \right] \quad \frac{e^{2-4}}{2}$

(iii) $v \geq 0$

$$f_U(u) = \int_0^\infty f_{UV}(u,v)\,dv$$

88 $$= \int_0^\infty e^{-(u+2v)}\,dv.$$

$$= -\frac{e^{-u}}{2}\left[e^{-2v}\right]_0^\infty$$

$$= -\frac{e^{-u}}{2}[0-1]$$

$$= \frac{e^{-u}}{2}.$$

$$= f_U(u) \quad \text{--- (2)}$$

Expected values of 2D = R.V.

If (X, Y) is a 2D discrete R.V with joint PRP mass func ~~fo.~~ P_{ij} then

$$E[g(x,y)] = \sum_i \sum_j g(x,y)\,P_{ij}$$

where $P_{ij} = P(X = x_i, Y = y_j)$

- If (X,Y) is a 2D continuous R.V with joint prp density func as $f(x,y)$ then

$$E[g(x,y)] = \int_x \int_y g(x,y)\,f(x,y)\,dx\,dy.$$

Properties of Expected Values of 2D R.V:

1) $E[g(X)] = \int_{-\infty}^{\infty} g(x)\,f_x(x)\,dx$

proof: $E[g(x)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)\cdot f(x,y)\,dx\,dy$ where $f_x(x)$ is marginal

Then, func. of X.

$$E[g(x)] = \int_{-\infty}^{\infty} g(x)\left[\int_{-\infty}^{\infty} f(x,y)\,dy\right]dx.$$

$$= \int_{-\infty}^{\infty} g(x)\cdot f_x(x)\,dx.$$

2) $E[h(Y)] = \int_{-\infty}^{\infty} h(y)\,f_y(y)\,dy$

where $f_y(y)$ is marginal

Then, func of Y

Proof: $E[h(Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(y)\,f(x,y)\,dx\,dy$

$$= \int_{-\infty}^{\infty} h(y)\left[\int_{-\infty}^{\infty} f(x,y)\,dx\right]dy$$

$$E[h(y)] = \int_{-\infty}^{\infty} h(y)\, f_y(y)\, dy \qquad \because \int_{-\infty}^{\infty} f(x,y)\,dx = f_y(y)$$

3) If X, Y are 2 R.V then the mathematical expectation of sum of these two R.V is equal to sum of their individual expectations $E[X+Y] = E(X) + E(Y)$

89

Proof: $E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)\, f(x,y)\, dx\, dy$.

$= E(X) + E(Y)$

hence proved.

4) $E[XY] \neq E[X]E[Y]$. But if X and Y are independent R.V. Then $E[XY] = E[X]E[Y]$.

Proof: $E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$

If X and Y are independent

$f(x,y) = f_X(x) \times f_y(y)$

$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x) \cdot f_y(y)\, dx\, dy$.

$= \int_{-\infty}^{\infty} x \cdot f_x(x)\, dx \int_{-\infty}^{\infty} y\, f_y(y)\, dy$.

$= E[X] \cdot E[Y]$

$\Rightarrow E[XY] = E[X] \cdot E[Y]$.

<u>Variance</u>: Let X be a R.V and $E[X]$ be the expectation of X then $E[X - E(X)]^2$ is defined as variance of X and is denoted by $Var(X)$

Proof: $Var(X) = E[X - E(X)]^2$

$= E[X^2 + E(X)^2 - 2X E(X)]$

We know that

$E[A+B] = E[A] + E[B]$

$Var(X) = E[X^2] + [E(X)]^2 - E[2X E(X)]$

$= E[X^2] + [E(X)]^2 - E(2X) E(X)$

$\therefore E(\text{const} \times RV) = \text{const}\, E(RV)$.

$Var(X) = E[X^2] + [E(X)]^2 - 2E(X) \cdot E(X)$

$= E(X^2) + [E(X)]^2 - 2[E(X)]^2$

$= E[X^2] - [E(X)]^2$

1) If a is a const. and X is a R.V then $Var(aX) = a^2 Var(X)$

Proof: $Var(X) = E[X - E(X)]^2$

90 $Var(aX) = E[aX - E(aX)]^2$

$= E[aX - aE(X)]^2$

$= a^2 E[X - E(X)]^2$

$= a^2 Var(X)$

2) If b is a const and X is a R.V then $Var(X+b) = Var(X)$

Proof: $Var(X+b) = E[(X+b) - E(X+b)]^2$

$= E[(X+b) - E(X) - E(b)]^2$

$= E[X + b - E(X) - b]^2$

$= E[X - E(X)]^2$

3) If X and Y are random variables and a and b are const. then, $Var(aX+bY) = a^2 Var(X) + b^2 Var(Y) - 2ab\, cov(X,Y)$

Proof: $Var(aX+bY) = E[(aX+bY) - E(aX+bY)]^2$

$\because Var(X) = E[X - E(X)]^2$

$Var(aX, bY) = E[(aX+bY) - E(aX) - E(bY)]^2$

$= E[aX + bY - aE(X) - bE(Y)]^2 = E[a\{X - E(X)\} + b\{Y - E(Y)\}]$

$= E[a^2\{X - E(X)\}^2 + b^2\{Y - E(Y)\}^2 + 2ab\{X - E(X)\}\{Y - E(Y)\}]$

$= a^2 E\{X - E(X)\}^2 + b^2 E\{Y - E(Y)\}^2 + 2ab E[\{X - E(X)\}\{Y - E(Y)\}]$

$= a^2 Var(X) + b^2 Var(Y) + 2ab\, cov(X,Y)$

Co-variance: If X, Y are 2D - R.V then the co-variance coff of co-relation of X and Y is denoted by $cov(X,Y)$ or C_{xy} is defined as degree or extent to which variables of a bivarient distribution are related with each other.

- coff of co-relation is a measure of dependence b/w variables X and Y

$$cov(X,Y) = E[(X - \mu_x)(Y - \mu_y)] = E[\{X - E(X)\}\{Y - E(Y)\}]$$

$$\rho_{xy} = \frac{cov(X,Y)}{\sigma_x \sigma_y}$$

Properties :

91

1) $\operatorname{cov}(X,Y) = E[XY] - E[X]\,E[Y]$

Proof: $\operatorname{cov}(X,Y) = E[(X-\mu_x)(Y-\mu_y)]$

$$= E[XY - X\mu_y - Y\mu_x + \mu_x\mu_y]$$
$$= E[XY] - E[X\mu_y] - E[Y\mu_x] + E[\mu_x\mu_y]$$
$$= E[XY] - E[X]\,E[Y] - \cancel{E[Y]\,E[X]} + \cancel{E[X]\,E[Y]}$$
$$= E[XY] - E[X]\,E[Y]$$

2) $|\rho_{xy}| \leq 1 \qquad (-1 \leq \rho_{xy} \leq 1)$

proof: $E[a(X-\mu_x)+(Y-\mu_y)]^2 = E[a^2(X-\mu_x)^2+(Y-\mu_y)^2+ 2a(X-\mu_x)(Y-\mu_y)]$

$$= E[a^2(X-\mu_x)^2] + E[(Y-\mu_y)^2] + 2E[a(X-\mu_x)(Y-\mu_y)]$$
$$= a^2 E[(X-\mu_x)^2] + E[Y-\mu_y]^2 + 2aE[(X-\mu_x)(Y-\mu_y)]$$

$\mu_x = E[X] \quad \mu_y = E[Y]$

$$= a^2 E[X-E(X)]^2 + E[Y-E(Y)]^2 + 2aE[\{X-E(X)\}\{Y-E(Y)\}]$$
$$= a^2 \operatorname{Var}(X) + \operatorname{Var}(Y) + 2a\operatorname{cov}(X,Y)$$
$$= ax^2 + c + bx$$

$b^2 - 4ac < 0$

$\because a^2\operatorname{Var}(X) + \operatorname{Var}(Y) + 2a\operatorname{cov}(X,Y)$

This eqn is quadratic exp in a.

$$[(2\operatorname{cov}(X,Y))^2 - 4\operatorname{Var}(X)\operatorname{Var}(Y)] < 0$$
$$[\operatorname{cov}(X,Y)]^2 < 4\operatorname{Var}(X)\operatorname{Var}(Y).$$
$$[\operatorname{cov}(X,Y)]^2 < \sigma_x^2\sigma_y^2$$
$$|\operatorname{cov}(X,Y)| < |\sigma_x\sigma_y|$$
$$\rho_{xy} = \frac{\operatorname{cov}(X,Y)}{\sigma_x\sigma_y}$$
$$\left|\frac{\operatorname{cov}(X,Y)}{\sigma_x\sigma_y}\right| \leq 1$$
$$|\rho_{xy}| \leq 1$$

Note: 92

1) If the coeff of co-relation = 1, then the R.V are said to be positively co-relation ted.

2) If the coeff of co-relation = -1, Then the R.V are said to be negatively co-related.

3) If coeff of co-relation = 0, then the R.V are said to be un-related.

4) If X and Y are independent R.V, then co-variance of X, Y = 0

Proof: $\text{cov}(X, Y) = E[XY] - E[X]E[Y]$
$$= E[X]E[Y] - E[X]E[Y]$$
$$= 0.$$

5) For independent R.V, co-relation coeff is also 0.
$$\rho_{xy} = \frac{\text{cov}(x,y)}{\sigma_x \sigma_y} = \frac{0}{\sigma_x \sigma_y} = 0.$$

3) If X, Y are R.V and a, b are constants then.
$$\text{cov}(aX, bY) = ab\,\text{cov}(X, Y)$$

Proof: $\text{cov}(X, Y) = E[(X-\mu_x)(Y-\mu_y)]$
$$\text{cov}(aX, bY) = E[\{aX - E(aX)\}\{bY - E(bY)\}]$$
$$= E[\{aX - aE(X)\}\{bY - bE(Y)\}]$$
$$= ab\,E[\{X - E(X)\}\{Y - E(Y)\}]$$
$$\text{cov}(aX, bY) = ab\,\text{cov}(X, Y)$$

Q) For the given joint probability distribution of 2 R.V. Find (i) Mean of X and Y
(ii) Var(X)
(iii) Var(Y)
(iv) cov(X, Y)

X \ Y	-1	0	1
-1	0	0	1/3
0	0	0	0
1	0	1/3	1/3

sol (i) $\sum_{x_i=-1}^{1} \sum_{y_j=-1}^{1} x_i P(X=x_i, Y=y_i)$

$$E(X) = \sum_{y_j=-1}^{1} (-1) P(X=-1, Y=y_j) + \sum_{y_j=-1}^{1} (0) P(X=0, Y=y_j) + \sum_{y_j=-1}^{1} (1) P(X=1, Y=y_j)$$

$$= (-1)(0+0+0) + 0(0+0+1/3) + (1)\left(\frac{1}{3}+0+\frac{1}{3}\right)$$

$$= \frac{2}{3}$$

$$E(Y) = \sum_{x_i} \sum_{y_j} P(X=x_i, Y=y_j)$$

$$= \sum_{x_i=-1}^{1} (-1) P(X=x_i, Y=-1) + \sum_{x_i} (0) P(X=x_i, Y=0) + \sum_{x_i} (1) P(X=x_i, Y=1)$$

$$= (-1)(0+0+1/3) + 0 + 1(0+1/3+1/3)$$

$$= \frac{1}{3}$$

(ii) $Var(X) = E[X^2] - [E(X)]^2$

$$E[X^2] = \sum_{x_i} \sum_{y_j} x_i^2 P(X=x_i, Y=y_j)$$

$$= \sum_{y_j} (-1)^2 P(X=-1, Y=y_j) + \sum_{y_j} (0)^2 P(X=0, Y=y_j) + \sum_{y_j} (1)^2 P(X=1, Y=y_j).$$

$$= 1(0+0+0) + 0(\ ) + (1)\left(\frac{1}{3}+0+\frac{1}{3}\right)$$

$$= \frac{2}{3}$$

$$Var(X) = \frac{2}{3} - \left(\frac{2}{3}\right)^2$$

$$= \frac{2}{9}$$

(iii) $Var(Y) = E(Y) - [E(Y)]^2$

$$E(Y^2) = \sum_{x_i=-1}^{1} \sum_{y_j=-1}^{1} y^2 P(X=x_i, Y=y_i)$$

$$= \sum_{x_i} (-1)^2 P(X=x_i, Y=-1) + (0)^2 P(X=x_i, Y=y_0) + \sum_{x_i} (1)^2 P(X=x_i, Y=1)$$

$$= (1)(0+0+1/3) + 0 + (1)(0+1/3+1/3)$$

$$= 1$$

$$E[Y^2] = 1$$

$$Var(Y) = 1 - \left(\frac{1}{3}\right)^2 = \frac{8}{9}$$

$\frac{A^2}{4\pi}[\sin(2\omega_0 t + \omega_0 T) - \sin(2\omega_0 t + \omega_0 T)] + \frac{A^2}{2}\cos\omega_0 T +$

$\frac{A}{2\pi}[-\sin(\omega_0 t + \omega_0 \tau) + \sin(\omega_0 t + \omega_0 T)]$.

(iv) $cov(X,Y) = E[XY] - E[X]E[Y]$

$E[XY] = \sum_{x_i}\sum_{y_j} xy\, P(X = x_i, Y = y_j)$

$= \sum_{x_i} x(-1)P(X = x_i, Y = -1) + \sum_{x_i}(0)\, P(X = x_i, Y = 0) + \sum_{x_i} x_i(1)P(X = x_i, Y = 1)$

$= (-1)[(-1)0 + 0(0) + 1(1/3)] + 0 + [(-1)(0) + (0)(1/3) + (1)(1/3)]$

$= -1/3 + 1/3$

$= 0$

$cov(X,Y) = 0 - \frac{2}{3}\left(\frac{1}{3}\right)$

$= \frac{-2}{9}$

Q) The joint pdf is given by $f(x) = \begin{cases} 2(x+y-2xy) \\ ;0 \le x \le 1 \\ ;0 \le y \le 1 \end{cases}$
find
(i) Var(x) (ii) Var(y) (iii) cov(x,y)

Sol (i) $E(X) = \int_{x=0}^{1} x f_X(x)\,dx$.

$f_X(x) = \int_{y=0}^{1} f(x,y)\,dy = \int_0^1 (2x + 2y - 4xy)\,dy$.

$= \left[2xy + \frac{2y^2}{2} - 4x\frac{y^2}{2}\right]_0^1$

$= (2x + 1 - 2x)$

$= 1$

$E(X) = \int_{x=0}^{1} x f_X(x)\,dx = \int_{x=0}^{1} x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$

$E(X^2) = \int_{x=0}^{1} x^2 f_X(x)\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

$Var(X) = E(X^2) - [E(X)]^2$

$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$