Classification of Random Processes

→ Discrete Random Sequence
→ Discrete Random Process
→ Continuous Random Sequence
→ Continuous Random Process
→ Deterministic Random Process
→ Non Deterministic Random Process

1) Discrete Random Sequence

If both T and S are discrete, the random process is called Discrete Random Sequence.

Eg :- X_n represents the outcome of nth toss of a fair dice then $\{X_n, n \geq 1\}$ is a discrete random sequence, since $T = \{1, 2, 3 ---\}$ & $S = \{1, 2, 3, 4, 5, 6\}$

2) Discrete Random Process

If 'T' is continuous and S is discrete, the random process is called Discrete Random Process.

Eg: X_t represents number of telephone calls received in the interval $(0, t)$ then $\{X(t)\}$ is a discrete random process since $S = \{0, 1, 2, 3 ---\}$

3) <u>Continuous Random Sequence</u>

If T is discrete and S is continuous, the random process is called a continuous Random Sequence.

Eg: X_n represents the temperature at the end of nth hour of the day then $\{X_n, 1 \leq n \leq 24\}$ is a Continuous Random Sequence since temperature can take any value in an interval and hence continuous

4) <u>Continuous Random process</u>

If both T and S are continuous, the random process is called Continuous Random process.

Eg: If $X(t)$ represents maximum temperature at a place in the interval $(0,t)$, $\{X(t)\}$ is a continuous Random process.

Note :- Word 'Discrete' or 'Continuous' is used to refer to the nature of 'S'.

Word 'Sequence' or 'process' is used to refer to the nature of T.

5) <u>Deterministic Random Process</u>

If the future value of any sample function can be predicted from the knowledge of the past value, then the process is called as Deterministic Random process.

6) <u>Non-Deterministic Random Process</u>

If the future value of any sample function cannot be predicted from the knowledge of past value, then the process is called Non-Deterministic Random process.

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Average values of Random Processes
or
Statistical averages of Random processes

(1) **Mean**

Mean of the process $\{X(t)\}$ is the expected value of a typical member $X(t)$ of the process.

$$\mu(t) = E\{X(t)\}$$

$$E\{X(t)\} = \int_{-\infty}^{\infty} x f(x,t)\, dx$$

(2) **Auto - Correlation**

Autocorrelation of the process $\{X(t)\}$ denoted by $R_{XX}(t_1, t_2)$ or $R_X(t_1, t_2)$ or $R(t_1, t_2)$ is the expected value of the product of any two members $X(t_1)$ and $X(t_2)$ of the process.

$$R(t_1, t_2) = E\{X(t_1) \times X(t_2)\}$$

3) **Autocovariance**

Autocovariance of the process $\{X(t)\}$, denoted by $C_{XX}(t_1, t_2)$ or $C_X(t_1, t_2)$ or $C(t_1, t_2)$ is defined as

$$C(t_1, t_2) = E\left[\{X(t_1) - \mu(t_1)\}\{X(t_2) - \mu(t_2)\}\right]$$

$$= R(t_1, t_2) - \mu(t_1) \times \mu(t_2)$$

$$= E\{X(t_1) \times X(t_2)\} - \mu(t_1) \times \mu(t_2)$$

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(4) **Correlation Coeffecient**

Correlation Coeffecient of the process denoted by $\rho_{xx}(t_1,t_2)$ or $\rho(t_1,t_2)$ is defined as

$$\rho(t_1,t_2) = \frac{C(t_1,t_2)}{\sqrt{C(t_1,t_1)\,C(t_2,t_2)}}$$

Where $C(t_1,t_1)$ is the variance of $X(t_1)$

* When we deal with two or more random processes, we can use joint distribution functions or averages to describe the relationship between them.

(5) **Cross - Correlation of 2 processes**

Cross - correlation of 2 processes $\{X(t)\}$ and $\{Y(t)\}$ is defined as

$$R_{xy}(t_1,t_2) = E\{X(t_1) \times Y(t_2)\}$$

(6) **Cross - Covariance**

Cross - covariance of 2 processes $\{X(t)\}$ and $\{Y(t)\}$ is defined as

$$C_{xy}(t_1,t_2) = R_{xy}(t_1,t_2) - \mu_x(t_1) \times \mu_y(t_2)$$

(7) **Cross - Correlation Coeffecient**

Cross correlation coeffecient of 2 processes $\{X(t)\}$ and $\{Y(t)\}$ is defined as

$$\rho_{xy}(t_1,t_2) = \frac{C_{xy}(t_1,t_2)}{\sqrt{C_{xx}(t_1,t_1) \times C_{yy}(t_2,t_2)}}$$

STATIONARITY | Stationary Random Process

A random process is called a strongly stationary process or strict sense stationary process (SSS) if all its finite dimensional distributions are invariant under translation of time period (or shift in time eg:- $X(t)$ and $X(t+T)$ possesses same tic properties i.e if the joint distributions (Joint density) of $X(t_1), X(t_2) ---- X(t_m)$ is the same as that of $X(t_1+h), X(t_2+h), X(t_3+h) --- X(t_m+h)$ for all $t_1, t_2, ---- t_m$ and $h(>0)$ and for all $m \geq 1$, then the random process is called SSS process.

First order Stationary Random process

A random process is said to be first order stationary if its first order density function is invariant under shift in time (translation in time) i.e densities of $X(t)$ and $X(t+h)$ are same

$$f(x,t) = f(x,t+h)$$

This is possible if $f(x,t)$ is independent of t

$\therefore E\{X(t)\}$ is also independent of t

$\mu =$ constant

Second order Stationary process

A random process is said to be second order stationary if the joint probability density functions (pdf) of $\{X_1(t_1)$ and $\{X(t_2)\}$ doesnot change with shift (translation) in time.

Joint pdf of $\{X(t_1), X(t_2)\}$ is same as $\{X(t_1+h), X(t_2+h)\}$

$$f(x_1, x_2, t_1, t_2) = f(x_1, x_2, t_1+h, t_2+h)$$

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This is possible only if $f(x_1, x_2, t_1, t_2)$ is a function of $t = t_1 - t_2$

WIDE SENSE STATIONARITY (WSS)

A random process $\{X(t)\}$ with finite first order and second order moments is called a WEAKLY STATIONARY PROCESS OR COVARIANCE STATIONARY PROCESS OR WIDE SENSE STATIONARY PROCESS (WSS) if

(i) Its mean is constant

$$E\{X(t)\} = \mu = \text{constant}$$

(ii) Autocorrelation depends only on time difference.

$$E\{X(t) \times X(t-\tau)\} = R(\tau)$$

★ SSS process with finite first and second order moments is a WSS while a WSS process need not to be a SSS process.

★ A random process that is not stationary in any sense is called an Evolutionary process.

★ Two random processes $\{X(t)\}$ and $\{Y(t)\}$ are said to be Jointly Stationary in the wide sense, if each process is individually a WSS process and $R_{XY}(t_1, t_2)$ is a function of $(t_1 - t_2)$ only.

Q:- The process $\{X(t)\}$ whose probability distribution under certain conditions is given by

$$P[\{X(t)\} = n] = \frac{(at)^{n-1}}{(1+at)^{n+1}}, \quad n = 1, 2, 3 \ldots$$

$$= \frac{at}{1+at}, \quad n = 0$$

Show that it is not stationary.

Sol:- For a process to be stationary:-

(i) Mean $E[\{X(t)\}]$ is a constant -

(ii) $E[\{X^2(t)\}]$ is constant -

(iii) $R_{XX}(t_1, t_2)$ is a function of $(t_1 - t_2)$

(iv) $Var\{X(t)\}$ is a constant.

The probability distribution of $X(t)$ is

$X(t) = n$	0	1	2	3	$\ldots\ldots$
p_n	$\frac{at}{1+at}$	$\frac{1}{(1+at)^2}$	$\frac{at}{(1+at)^3}$	$\frac{(at)^2}{(1+at)^4}$	

$$E\{X(t)\} = \sum_{n=0}^{\infty} n\, p_n$$

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$$E\{X(t)\} = \frac{1}{(1+at)^2} + \frac{2at}{(1+at)^3} + \frac{3(at)^2}{(1+at)^4} + \cdots \infty$$

$$= \frac{1}{(1+at)^2}\left[1 + \frac{2at}{1+at} + \frac{3(at)^2}{(1+at)^2} + \cdots\right]$$

Put $\frac{at}{1+at} = x$

$$= \frac{1}{(1+at)^2}\left[1 + 2x + 3x^2 + \cdots\right]$$

$$= \frac{1}{(1+at)^2} \cdot (1-x)^{-2}$$

$$= \frac{1}{(1+at)^2}\left[1 - \frac{at}{1+at}\right]^{-2} = \frac{1}{(1+at)^2}\left(\frac{1+at-at}{1+at}\right)^{-2}$$

$$\boxed{E\{X(t)\} = 1}$$

$$E\{X^2(t)\} = \sum_{n=0}^{\infty} n^2 p_n = 0 + \sum_{n=1}^{\infty} n^2 p_n$$

$$= \sum_{n=1}^{\infty} n(n+1) p_n - \sum_{n=1}^{\infty} n p_n$$

$$= \sum_{n=1}^{\infty} n(n+1) \frac{(at)^{n-1}}{(1+at)^{n+1}} - 1$$

$\because \sum_{n=1}^{\infty} n p_n = \text{Mean}$

$$E\{X^2(t)\} = \left[\frac{2}{(1+at)^2} + 2(3)\frac{at}{(1+at)^3} + 3(4)\frac{(at)^2}{(1+at)^4} + \cdots \infty\right] - 1$$

$$= \frac{2}{(1+at)^2}\left[1 + \frac{3at}{1+at} + \frac{6(at)^2}{(1+at)^2} + \cdots \infty\right] - 1$$

$$= \frac{2}{(1+at)^2}\left[1 + 3x + 6x^2 + \cdots \infty\right] - 1$$

where $x = \dfrac{at}{1+at}$

$$= \frac{2}{(1+at)^2}(1-x)^{-3} - 1$$

$$= \frac{2}{(1+at)^2}\left[1 - \frac{at}{1+at}\right]^{-3} - 1$$

$$= \frac{2}{(1+at)^2}(1+at)^3 - 1$$

$$E\{X^2(t)\} = 2(1+at) - 1 = 2 + 2at - 1 = \underline{\underline{1+2at}}$$

$$\boxed{E\{X^2(t)\} = 1 + 2at}$$

$$\mathrm{Var}[X(t)] = E\{X^2(t)\} - [E(X(t))]^2 = 1 + 2at - 1 = 2at$$

$$\boxed{\mathrm{Var}[X(t)] = 2at}$$

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Since $E\{X^2(t)\}$ is not a constant; it is function of t

and since $Var\{X(t)\}$ is not a constant, it is function of t

$\therefore$ Given process is not stationary

Unit - IV

Random Processes

- A random variable is a fn which assignes a real no. to every outcome of a random exp. while a random process is a fn that assigns a time fn to every outcome of a random exp.
- Set of fns $\{x_1(t), x_2(t) \dots x_6(t)\}$ represents a random process.

Classification :

1) Discrete Random Sequence
2) Discrete Random process
3) Continuous Random Sequence
4) Continuous Random process
5) Deterministic Random process
6) Non-deterministic Random process

Statistical Averages of Random processes :

1) Mean: $\mu(t) = E[X(t)] = \int_{-\infty}^{\infty} x(t) \cdot f(x,t)\, dx$.

2) Auto co-relation : Auto-corelation of a process $[x(t)]$ is denoted by $R_{XX}(t_1,t_2)$, $R_X(t_1,t_2)$ or $R(t_1,t_2)$

$$R_{XX}(t_1,t_2) = E[\{x(t_1)\}\{x(t_2)\}]$$

$$x(t) = A \sin \lambda t$$

3) Auto-covariance : Auto co-variance of a process $\{x(t)\}$ is denoted by $C_{XX}(t_1,t_2)$ or $C_X(t_1,t_2)$ or $C(t_1,t_2)$.

$$C_{XX}(t_1,t_2) = E[\{x(t_1) - \mu(t_1)\}\{x(t_2) - \mu(t_2)\}]$$

$$C_{XX}(t_1,t_2) = E[\{x(t_1)\}\{x(t_2)\}] - \mu(t_1)\mu(t_2)$$

4) Co-relation coefficient :

$$\rho(t_1,t_2) = \frac{C_{XX}(t_1,t_2)}{\sqrt{C(t_1,t_1)\, C(t_2,t_2)}}$$

$C(t_1,t_1)$ = Variance of $X(t_1)$

$C(t_2,t_2)$ = Variance of $X(t_2)$

5) Cross co-relation: of $\{x(t)\}$ and $\{y(t)\}$ is denoted by $R_{xy}(t_1, t_2)$

$$R_{xy}(t_1, t_2) = E[\{x(t_1)\}\{y(t_2)\}]$$

6) cross co-variance: of $\{x(t)\}$ and $\{y(t)\}$ is dentoted by $C_{xy}(t_1, t_2)$

$$C_{xy}(t_1, t_2) = R_{xy}(t_1, t_2) - \mu_x(t_1)\mu_y(t_2)$$

7) Cross-co-relation coefficient:

$$\rho_{xy}(t_1, t_2) = \frac{C_{xy}(t_1, t_2)}{\sqrt{C_x(t_1, t_1)\, C_y(t_2, t_2)}}$$

$C_x(t_1, t_1)$ = variance of $X(t_1)$
$C_y(t_2, t_2)$ = variance of $X(t_2)$

Stationarity / Stationary Random Process: A random process is called a strongly stationary process or strict sense stationary (SSS) if all its distributions are invarient under translation of time.

First order Stationary Random process: A RP is said to be first order stationary if its density func. is invarient under the translation in time i.e

$$f(x, t) = f(x, t+h)$$

Second order Stationary Random process: A RP is said to be second order stationary if the joint pdf of $\{x(t_1)\}$ and $\{x(t_2)\}$ are invarient under translation i.e

$$f(x_1, x_2, t_1, t_2) = f(x_1, x_2, t_1+h, t_2+h)$$

* Wide Sense Stationarity (WSS): A RP $\{x(t)\}$ is called a wide sense stationary process if

(i) $E[\{x(t)\}]$ = constant

(ii) Auto co-relation depends only on time defference

$$E[x(t)\, x(t-\tau)] = R(\tau)$$

$R_{XX}(t_1,t_2)$ is a fn of time difference (t_1-t_2)

Note:

1) A strict sense SP is a WSS process but WSS process need not be a SSS process.

2) A RP that is not stationary in any sense whether in strict sense or in wide sense.

3) Two RP $\{X(t)\}$ $\{Y(t)\}$ are said to be jointly stationary in wide sense if each process is individually a WSS process and cross co-relation $R_{XY}(t_1,t_2)$ is a fn of t_1-t_2 only.

Q) Show that RP $x(t)=A\cos(\omega_0 t+\theta)$ is WSS if A and ω_0 are constants and θ is uniformly distributed in the interval $(0,2\pi)$

sol For a process to be WSS

(i) $E[x\{t\}]=$ const.

(ii) $E[x(t)\,x(t-\tau)]=R(\tau)$.

$R(t_1,t_2)$ is a fn of (t_1-t_2) given is uniformly distributed R.V $f(\theta)=\begin{cases} 1/b-a & ; a<\theta<b \\ 0 & ; o/w \end{cases}$

$$f(\theta)=\frac{1}{2\pi}\ ,\ 0<\theta<2\pi.$$

(i) $E[x(t)]=$ const

$$E[x(t)]=\int_0^{2\pi} x(t)f(\theta)d\theta.$$

$$=\frac{1}{2\pi}\int_0^{2\pi} A\cos(\omega_0 t+\theta)d\theta.$$

$$=\frac{A}{2\pi}\left[\sin(\omega_0 t+\theta)\right]_0^{2\pi}$$

$$=\frac{A}{2\pi}\left[\sin(2\pi+\omega_0 t)-\sin\omega_0 t\right]$$

$$=\frac{A}{2\pi}\left[\sin\omega_0 t-\sin\omega_0 t\right]$$

$$=0\text{(const)}$$

(ii) $R_{XX}(t_1, t_2) = E[x(t_1)\, x(t_2)]$

106 $x(t_1) = A\cos(\omega_0 t_1 + \theta)$

$x(t_2) = A\cos(\omega_0 t_2 + \theta)$

$$R_{XX}(t_1,t_2) = E[A\cos(\omega_0 t_1+\theta) \times A\cos(\omega_0 t_2+\theta)]$$

$$= E\left[\frac{A^2}{2} 2\cos(\omega_0 t_1+\theta) \times \cos(\omega_0 t_2+\theta)\right.$$

$$= E\left[\frac{A^2}{2}[\cos[\omega_0(t_1+t_2)+2\theta] + \cos(\omega_0(t_1-t_2))]\right]$$

$$= \frac{A^2}{2} E[\cos\{\omega_0(t_1+t_2)+2\theta\} + \cos\omega_0(t_1-t_2)]$$

$$E[X+Y] = E[X] + E[Y]$$

$$R_{XX}(t_1,t_2) = \frac{A^2}{2} E[\cos(\omega_0(t_1+t_2)+2\theta)] + \frac{A^2}{2} E[\cos(\omega_0(t_1-t_2))]$$

$$= \frac{A^2}{2}\int_0^{2\pi} \cos[\omega_0(t_1+t_2)+2\theta] f(\theta)\,d\theta + \frac{A^2}{2}\int_0^{2\pi} \cos(\omega_0(t_1-t_2)) f(\theta)\,d\theta$$

$$= \frac{A^2}{4\pi}\left[\sin(\omega_0(t_1+t_2)) + 2\theta\right]_0^{2\pi} + \frac{A^2}{4\pi}\left[\theta\cos\omega_0(t_1-t_2)\right]_0^{2\pi}$$

$$R_{XX}(t_1,t_2) = \frac{A^2}{4\pi}\left[\sin[4\pi + \omega_0(t_1+t_2)] - \sin\omega_0(t_1+t_2)\right] + \frac{A^2}{4\pi}\cos\omega_0(t_1-t_2)[2\pi - 0]$$

$$= \frac{A^2}{4\pi}\left[\cancel{\sin(\omega_0(t_1+t_2))} - \sin\omega_0(t_1+t_2)\right] + \frac{A^2}{2}\cos\omega_0(t_1-t_2)$$

$$= \frac{A^2}{2}\cos\omega_0(t_1-t_2)$$

8) A R.P is defined as $X(t) = A\sin(\omega_0 t + \theta)$ where A and ω_0 are const if θ is uniformly distributed R.V in the interval $(-\pi, \pi)$. Show that $x(t)$ is WSS

Q) Show that the process $\{x(t)\} = A\cos\lambda t + B\sin\lambda t$, where A and B are R.V is a WSS process if (i) $E(A)=E(B)=0$ (ii) $E(A^2)=E(B^2)$ (iii) $E(AB)=0$. 107

Sol: For a process to be WSS $E[\{x(t)\}]=0$.

(a) $E[x(t)] = E[A\cos\lambda t + B\sin\lambda t]$

$= E[A\cos\lambda t] + E[B\sin\lambda t]$

$= \cos\lambda t \, E[A] + \sin\lambda t \, E[B]$ —— ①

For WSS process if $E(A)=E(B)$ then $E[x(t)]=0$ in ①

Hence $E(A)=E(B)=0$

$R_{xx}(t_1,t_2)$ must be fn of (t_1-t_2)

$R_{xx}(t_1,t_2) = E[\{x(t_1)\}\{x(t_2)\}]$

$x(t_1) = A\cos\lambda t_1 + B\sin\lambda t_1$

$x(t_2) = A\cos\lambda t_2 + B\sin\lambda t_2$

$R_{xx}(t_1,t_2) = E[(A\cos\lambda t_1 + B\sin\lambda t_1)(A\cos\lambda t_2 + B\sin\lambda t_2)]$

$= E[A^2\cos\lambda t_1\cos\lambda t_2 + B^2\sin\lambda t_1\sin\lambda t_2 + AB\sin(\lambda t_1+\lambda t_2)]$

$= \cos\lambda t_1\cos\lambda t_2 \, E(A^2) + \sin\lambda t_1\sin\lambda t_2 \, E(B^2) + \sin(\lambda t_1+\lambda t_2)\, E[AB]$

if $E(A^2)=E(B^2)=k$

$E[AB]=0$.

$R_{xx}(t_1,t_2) = k[\cos\lambda t_1\cos\lambda t_2 + \sin\lambda t_1\sin\lambda t_2]$

$= k\cos\lambda(t_1-t_2)$.

Auto. co-relation fn is func of (t_1-t_2)

Q) Two R.P's $X(t)$ and $Y(t)$ are given by $X(t) = A\cos\omega_0 t + B\sin\omega_0 t$, $Y = B\cos\omega_0 t - A\sin\omega_0 t$. Show that $X(t)$ and $Y(t)$ are jointly wide. Show that A, B are uncorrela R.V with zero mean and same variences and ω_0 is a const.

Sol! $cov(A, B) = E[AB] - E[A]E[B]$ $\qquad E(A^2) = E(B^2)$

108 $\mu_A = E(A) = 0$

$\mu_B = E(B) = 0$

$cov(A,B) = 0$

$0 = E[AB] - 0$

$E[AB] = 0$

$Var(A) = E[A^2] - [E(A)]^2$

$Var(A) = E(A^2)$

$Var(B) = E(B^2)$

$E(A^2) = E(B^2)$

(i) $E[x(t)] = E[A\cos\omega_0 t + B\sin\omega_0 t]$

$E[x(t)] = E[A\cos\omega_0 t] + E[B\sin\omega_0 t]$

$= \cos\omega_0 t \, E[A] + \sin\omega_0 t \, E[B]$

$\because E[A] = E[B] = 0$

$E[x(t)] = 0$

(ii) $R_{xx}(t_1, t_2) = E[\{x(t_1)\}\{x(t_2)\}]$

$= E[(A\cos\omega_0 t_1 + B\sin\omega_0 t_1)(A\cos\omega_0 t_2 + B\sin\omega_0 t_2)]$

$= E[A^2\cos\omega_0 t_1 \cos\omega_0 t_2 + B^2 \sin\omega_0 t_1 \sin\omega_0 t_2 +$

$AB\cos\omega_0 t_1 \sin\omega_0 t_2 + AB\sin\omega_0 t_1 \cos\omega_0 t_2$

$= \cos\omega_0 t_1 \cos\omega_0 t_2 \, E[A^2] + \sin\omega_0 t_1 \sin\omega_0 t_2 \, E[B^2]$

$= k[\cos\omega_0 t_1 \cos\omega_0 t_2 + \sin\omega_0 t_1 \sin\omega_0 t_2]$

$= k\cos\omega_0(t_1 - t_2)$

$\{x(t)\}$ is W.S.S

& $Y(t)$ is WSS..

(iii) $R_{xy}(t_1,t_2) = E[\{x(t_1)\}\{x(t_2)\}]$

$\{x(t_1)\} = A\cos\omega_0 t_1 + B\sin\omega_0 t_1$

$\{y(t_2)\} = B\cos\omega_0 t_2 - A\sin\omega_0 t_2$

$R_{xy}(t_1,t_2) = E[(A\cos\omega_0 t_1 + B\sin\omega_0 t_1)(B\cos\omega_0 t_2 - A\sin\omega_0 t_2)]$

$= E[AB\cos\omega_0 t_1 \cos\omega_0 t_2 - A^2\cos\omega_0 t_1 \sin\omega_0 t_2 + B^2\sin\omega_0 t_1 \cos\omega_0 t_2 - AB\sin\omega_0 t_1 \sin\omega_0 t_2]$

$= \cos\omega_0 t_1 \cos\omega_0 t_2\, E[AB] - \cos\omega_0 t_1 \sin\omega_0 t_2\, E(A^2) + \sin\omega_0 t_1 \cos\omega_0 t_2\, E(B^2) - \sin\omega_0 t_1 \sin\omega_0 t_2\, E(AB)$

$E(AB) = 0 \qquad E(A^2) = E(B^2) = k$

$R_{xy}(t_1,t_2) = k[\sin\omega_0 t_1 \cos\omega_0 t_2 - \cos\omega_0 t_1 \sin\omega_0 t_2]$

$= k[\sin(\omega_0 t_1 - \omega_0 t_2)]$

$= k\sin\omega_0(t_1 - t_2)$

<u>Ergodicity</u>:

<u>Time Average</u>: If $\{x(t)\}$ is the R.P Then $\frac{1}{2T}\int_{-T}^{T} x(t)\,dt$ is called the time avg of $\{x(t)\}$ over the interval $-T$ to T. It is denoted by μ.

<u>Ergodic Process</u>: A R.P is said to be ergodic if its statistical averages (ensemble) are equal to time averages.

<u>Mean ergodic process</u>: If a R.P $\{x(t)\}$ has a const. mean, $E[x(t)] = \mu$ and $\lim_{T\to\infty} \overline{x}_T = \mu$.

<u>Co-relation ergodic process</u>: The stationary process $\{x(t)\}$ is said to be co-relation ergodic if the process.

$Z(t) = \{x(t+\lambda)\,x(t)\}$ is mean ergodic and if

$\lim_{T\to\infty} \overline{Z}_T = R(\lambda)$

where $\overline{Z}_T = \frac{1}{2T}\int_{-T}^{T} x(t+\lambda)\,x(t)\,dt$.

$$R_{xx}(\lambda) = E[X(t+\lambda)X(t)]$$

110 $R_{xx}(t_1,t_2) = E[X(t_1)X(t_2)]$

$$Z_T = X(t+\lambda)X(t)$$

2) If WSS process $\{x(t)\}$ is given by $10\cos(100t+\theta)$ where θ is uniformly distributed over the interval $(-\pi,\pi)$. Prove that $\{x(t)\}$ is co-relation ergodic.

Sol $\lim_{T\to\infty} \bar{Z}_T = R(\lambda)$

$$R(\lambda) = E[X(t+\lambda)x(t)]$$

$$X(t+\lambda) = 10\cos[100(t+\lambda)+\theta]$$

$$R(\lambda) = E[10\cos\{100(t+\lambda)+\theta\}\,10\cos(100t+\theta)]$$

$$= E\left[\frac{100}{2} \times 2\cos(100t+100\lambda+\theta)\cos(100t+\theta)\right]$$

$$= E[50\cos(200t+200\lambda+2\theta) + 50\cos(100\lambda)]$$

$$= 50E[\cos(200t+200\lambda+2\theta)] + 50E[\cos(100\lambda)]$$

$$f(\theta) = \frac{1}{b-a};\ a<\theta<b$$

$$f(\theta) = \frac{1}{2\pi};\ -\pi<\theta<\pi$$

$$R(\lambda) = 50\int_{-\pi}^{\pi}\frac{1}{2\pi}\cos(200t+100\lambda+2\theta)\,d\theta + 50\int_{-\pi}^{\pi}\cos(100\lambda)\left(\frac{1}{2\pi}\right)d\theta$$

$$= \frac{50}{2\pi}\left[\sin(2\pi+200t+100\lambda) - \sin(-2\pi+200t+100\lambda)\right]$$
$$+ \frac{50}{2\pi}\cos(100\lambda)(2\pi)$$

$$\sin(-x) = -\sin x$$
$$\sin(2\pi-x) = -\sin x$$
$$\sin(2\pi+x) = \sin x.$$

$$R(\lambda) = \frac{25}{\pi}\left[\sin(200t+100\lambda) - \sin(200t+100\lambda)\right] + 50\cos 100\lambda$$

$$R(\lambda) = 50\cos(100\lambda)$$

$$\bar{Z}_T = \frac{1}{2T}\int_{-T}^{T} \lambda(t+\lambda)\, x(t)\, dt$$

$$= \frac{1}{2T}\int_{-T}^{T} 10\cos(100t+100\lambda+\theta)\, 10\cos(100t+\theta)\, dt$$

$$= \frac{1}{2T}\int_{-T}^{T} 50[\cos(200t+100\lambda+2\theta) + \cos(100\lambda)]\, dt$$

$$= \frac{25}{T}\int_{-T}^{T} \cos(200t+100\lambda+2\theta)\, dt + \frac{25}{T}\cos(100\lambda)\int_{-T}^{T} dt$$

$$= \frac{25}{T}\int_{-T}^{T} \cos(200t+100\lambda+2\theta) + 50\cos(100\lambda)$$

$$= 0 + 50\cos(100\lambda)$$

$$= R(\lambda)$$

Q) The process $\{x(t)\}$ whose probability distribution under certain cond. is given by

$$P[x(t)] = \frac{(at)^{n-1}}{(1+at)^{n+1}} \quad n = 1, 2, 3 \dots$$

$$= \frac{at}{1+at} \; ; n = 0$$

Sol Process to be SSS

(i) $E[X(t)]$ should be const.

(ii) $E[X^2(t)]$ should be const

(iii) $Var[x(t)]$ should be const.

(iv) $R_{XX}(t_1, t_2)$ should be fn $(t_1 - t_2)$

X_i	0	1	2	3	4 ...
$P[X(t)]$ $n=1,2,3..$	$\frac{at}{1+at}$	$\frac{1}{(1+at)^2}$	$\frac{at}{(1+at)^3}$	$\frac{(at)^2}{(1+at)^4}$	.....

(i) $E[X(t)] = \sum_{n=0}^{\infty} n\, P[X(t)]$

$$= \left(0 \times \frac{at}{1+at}\right) + \frac{1}{(1+at)} + \frac{2at}{(1+at)^3} + \frac{3(at)^2}{(1+at)^4} + \dots \infty$$

$$= \frac{1}{(1+at)^2}\left[1 + \frac{2at}{(1+at)} + \frac{3(at)^2}{(1+at)^2} + \dots \infty\right]$$

Let $\frac{at}{1+at} = x$.

$$E[X(t)] = \frac{1}{(1+at)^2}\left[1+2x+3x^2+4x^3+\dots\infty\right]$$

$$= \frac{1}{(1+at)^2}[1-x]^{-2}$$

$$= \frac{1}{(1+at)^2}\left[1-\frac{at}{1+at}\right]^{-2}$$

$$= \frac{1}{(1+at)^2}\left[\frac{1+at-at}{1+at}\right]^{-2}$$

$$= \frac{1}{(1+at)^2}\left(\frac{1}{1+at}\right)^{-2}$$

$$= 1$$

(ii) $$E[x^2(t)] = \sum_{n=0}^{\infty} n^2 P[X(t)]$$

$$= \sum_{n=0}^{\infty} n(n+1) P[X(t)] - \sum_{n=0}^{\infty} n P[X(t)]$$

Replace n^2 with $(n(n+1)-n)$

$$= \sum_{n=0}^{\infty} [n(n+1)-n] P[X(t)]$$

$$= \sum_{n=0}^{\infty} [n(n+1) P[X(t)] - n P[X(t)]]$$

$$= \left[\frac{2}{(1+at)^2} + \frac{2(3at)}{(1+at)^3} + \frac{3\times 4(at)^2}{(1+at)^4} + \dots\infty\right] - E(X(t))$$

$$= \frac{2}{(1+at)^2}\left[1+\frac{3at}{(1+at)} + 6\left[\frac{at}{1+at}\right]^2 + \dots\infty\right] - 1$$

$$= \frac{2}{(1+at)^2}\left[1+3x+6x^2+10x^3+\dots\infty\right] - 1$$

$$= \frac{2}{(1+at)}(1-x)^{-3} - 1$$

$$= \frac{2}{(1+at)}\left[1-\frac{at}{1+at}\right]^{-3} - 1$$

$$= \frac{2}{(1+at)}(1+at)^3 - 1$$

$$= 2(1+at) - 1$$

$$= 2at + 1$$

$E[x^2(t)]$ is not a const

Hence, this process is not a SSS process

(iii) $var[X(t)] = E[x^2(t)] - E[x(t)]^2$

$= 1 + 2at - 1$

$= 2at$

This process is not a stationary process.

<u>Power spectral density function</u>: If $\{x(t)\}$ is a stationary process either in strict sense or in wide sense with auto co-relation fn given as $R(\tau)$ then the fourier transforms of $R(\tau)$ is called the power spectral density fn of $\{x(t)\}$ denoted by $S_{xx}(\omega)$ or $S_x(\omega)$ or $S(\omega)$

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{i\omega\tau} d\tau \quad \text{—— (1)}$$

$$S(f) = \int_{-\infty}^{\infty} R(\tau) e^{-i2\pi f\tau} d\tau \quad \text{—— (2)}$$

<u>Weiner Khinchins Relation</u>: The auto corelation $R(\tau)$ is given by fourier inverse transform $S(\omega)$

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega \quad \text{—— (3)}$$

$$R(\tau) = \int_{-\infty}^{\infty} S(f) e^{i2\pi f\tau} df \quad \text{—— (4)}$$

- If $X(t)$ and $Y(t)$ are 2 jointly stationary processes with cross co-relation fn $R_{xx}(\tau)$. Then fourier transform of $R_{xy}(\tau)$ is called cross power spectral density of $\{x(t)\}$ and $\{y(t)\}$.

<u>Properties of power spectral density fn:</u>

1) The value of spectral density fn at zero frequency is equal to the total area under the graph of the auto-co-relation fn.

$$S(0) = \int_{-\infty}^{\infty} R(\tau) d\tau \quad \text{—— (5)}$$

$$S(0) = \int_{-\infty}^{\infty} R(\tau) d\tau \quad \text{—— (6)}$$

2) The mean square value of a WSS process is equal to the total area under the graph of spectral density

114 $E[X^2(t)] = R(0)$

put $\tau = 0$ in (4)

$$R(0) = \int_{-\infty}^{\infty} S(f)\,df.$$

3) The spectral density fn of a real R.P is an even fn

$$S(-\omega) = S(\omega)$$

4) The spectral density & auto co-relation fn of a real WSS process form a fourier cosine transform pair

Proof: $S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\tau$

$$= \int_{-\infty}^{\infty} R(\tau)[\cos\omega\tau - i\sin\omega\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} R(\tau)\cos\omega\tau - i\int_{-\infty}^{\infty} R(\tau)\underbrace{\sin\omega\tau}_{odd}\,d\tau$$

$$= 2\int_{0}^{\infty} R(\tau)\cos\omega\tau\,d\tau - 0.$$

$$= \int_{0}^{\infty} 2R(\tau)\cos\omega\tau\,d\tau.$$

$S(\omega)$ = fourier cosine transform of $2R(\tau)$

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, e^{i\omega\tau}\, d\omega.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)[\cos\omega\tau + i\sin\omega\tau)\,d\omega.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\cos\omega\tau\,d\omega + \frac{i}{2\pi}\int_{-\infty}^{\infty} S(\omega)\sin\omega\tau\,d\omega$$

$$= \frac{1}{\pi}\int_{0}^{\infty} S(\omega)\cos\omega\tau\,d\omega$$

3) **Wiener Khinchine Theorem**: If $X_T(\omega)$ is the fourier transform of RP defined as

$$X_T(t) = \begin{cases} x(t) & ; |t| \leq T \\ 0 & ; |t| > T \end{cases}$$

where $\{X_T(t)\}$ is a real WSS process.

Q) Find the power spectral density of a random telegraph signal with auto-correlation fn of the telegraph signal given by $R(T) = a^2 e^{-2y|T|}$

Sol.
$$S(\omega) = \int_{-\infty}^{\infty} R(T) e^{-i\omega T} dT$$

$$= \int_{-\infty}^{\infty} a^2 e^{-2y|T|}$$

$$= \int_{-\infty}^{0} a^2 e^{2yT} e^{-i\omega T} + \int_{0}^{\infty} a^2 e^{-2yT} e^{-i\omega T}$$

$$= a^2 \int_{-\infty}^{0} e^{+T(2y-i\omega)} dT + a^2 \int_{0}^{\infty} e^{-T(2y+i\omega} dT$$

$$= a^2 \left[\frac{e^{-T(2y-i\omega)}}{2y-i\omega}\right]_{-\infty}^{0} + a^2 \left[\frac{e^{-T(2y+i\omega)}}{2y+i\omega}\right]_{0}^{\infty}$$

$$= a^2 \left[\frac{1}{2y-i\omega}\right] - a^2 \left[\frac{-1}{2y+i\omega}\right]$$

$$= \frac{a^2}{2y-i\omega} + \frac{a^2}{2y+i\omega)} = \frac{4ya^2}{4y^2+\omega^2}$$

Q) Find the power spectral density of a WSS process with auto co-relation fn given as $R(T) = e^{-\alpha \tau^2}$

Sol.
$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\alpha\tau^2} e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{(-\alpha\tau^2 - i\omega\tau)} d\tau.$$

$$= \int_{-\infty}^{\infty} e^{-\alpha\left(T^2 + \frac{i\omega T}{\alpha}\right)} d\tau$$

Add & Sub $\frac{\omega^2}{4\alpha^2}$ to power of e

$$S(\omega) = \int_{-\infty}^{\infty} e^{-\alpha\left(T^2 + \frac{i\omega\tau}{\alpha} - \frac{\omega^2}{4\alpha^2} + \frac{\omega^2}{4\alpha^2}\right)} d\tau$$

11b $= \int_{-\infty}^{\infty} e^{-\alpha\left(\tau^2 + \frac{i\omega\tau}{\alpha} + \frac{\omega^2}{4\alpha^2}\right) - \alpha\left(\frac{\omega^2}{4\alpha^2}\right)} d\tau.$

$$= \int_{-\infty}^{\infty} e^{-\alpha\left(\tau + \frac{i\omega}{2\alpha}\right)^2} e^{-\alpha\left(\frac{\omega^2}{4\alpha^2}\right)} d\tau.$$

$$= e^{-\omega^2/4\alpha} \int_{-\infty}^{\infty} e^{-\alpha\left(\tau + \frac{i\omega}{2\alpha}\right)^2} d\tau$$

put $\sqrt{\alpha}\left(\tau + \frac{i\omega}{2\alpha}\right) = t$

$$\sqrt{\alpha}\, d\tau = dt$$

$$d\tau = \frac{dt}{2}$$

$$S(\omega) = e^{-\omega^2/4\alpha} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{\sqrt{\alpha}} \qquad e^{-t^2} = \text{even fn.}$$

$$= \frac{e^{-\omega^2/4\alpha}}{\sqrt{\alpha}} \cdot 2\int_0^{\infty} e^{-t}\, dt$$

put $t^2 = k$

$$2t\,dt = dk$$

$$S(\omega) = \frac{2e^{-\omega^2/4\alpha}}{\sqrt{\alpha}} \int_0^{\infty} e^{-k} \frac{dk}{2\sqrt{k}}$$

$$= \frac{e^{-\omega^2/4\alpha}}{\sqrt{\alpha}} \int_0^{\infty} e^{-k} k^{1/2 - 1}\, dk$$

$$= \frac{e^{-\omega^2/4\alpha}}{\sqrt{\alpha}} \Gamma(1/2) = \frac{e^{-\omega^2/4\alpha}}{\sqrt{\alpha}} \sqrt{\pi}$$

$$= \sqrt{\frac{\pi}{\alpha}}\, e^{-\omega^2/4\alpha}.$$