-3.2

3 ST

PROBABILITY AND RANDOM PROCESSES

Unit-I

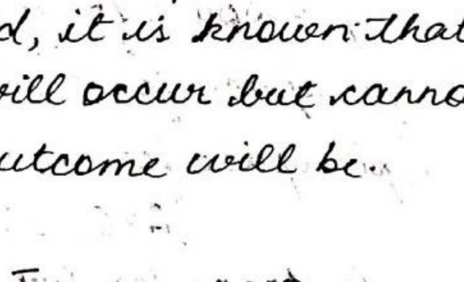
Deterministic theory: An experiment whose result can be predicted with maximum certainity is called a deterministic experiment.

Eg: If potential difference 'E' between two ends of the conductor and resistance 'R' are known, the current 'I' flowing through the conductor is uniquely determined by Ohm's law $I = \frac{V}{R} = \frac{E}{R}$

Non-deterministic experiment / Random Experiment: An experiment whose result cannot be predicted with max. certainity and there is a chance of an event may or may not occuring, then the experiment is called Non-deterministic or random experiment.

Eg: If a 6 face cubic dice is rolled, it is known that any of the 6 possible outcomes will occur but cannot be predicted what exactly the outcome will be.

Properties:



1. Commutative law:

$$A \cup B = B \cup A$$

2. Associative law:

$$(A \cup B) \cup C = A \cup (B \cup C)$$

3. Distributive law:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

4. Demorgans law:

$$\overline{A \cup B} = \cancel{\bar{A} \cup \bar{B}}\ \bar{A} \cap \bar{B}$$

Properties:

1. commutative law

$$A \cap B = B \cap A$$

2. Associative law

$$(A \cap B) \cap C = A \cap (B \cap C)$$

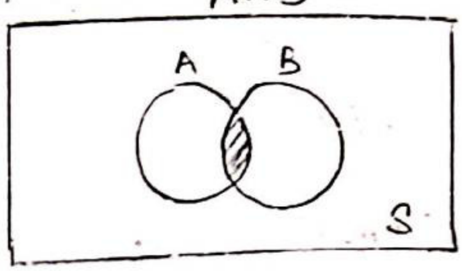
3. Distributive law

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

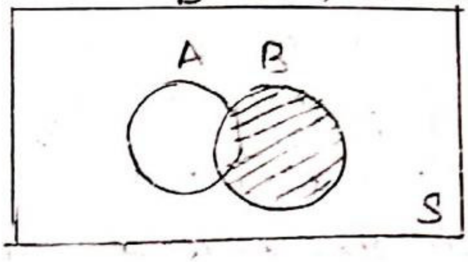
4. Demorgans law

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

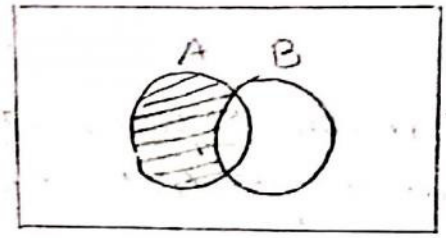
II. $A \cap B$



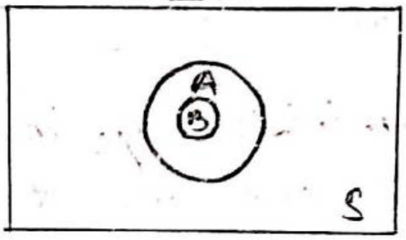
III. $B - A \Rightarrow \bar{A} \cap B$



IV. $A - B \Rightarrow A \cap \bar{B}$



V. $B \subseteq A$



Sample Space: The set of all possible outcomes of an experiment is called ~~sp~~ sample space denoted by 'S'.

Eg: Sample Space for a coin tossed $S = \{head, tail\}$

Sample space for a dice rolled $S = \{1, 2, 3, 4, 5, 6\}$

Sample Space for two dice rolled $S = \{(1,1)(1,2) \ldots (6,6)\}$

$6 \times 6 = 36$ elements.

Sample Space for two coin tossed $S = \{(H,T), (T,H) (H,H) (T,T)\}$

Event A: head comes in first toss $A = \{HH, HT\}$

Event B: For getting atleast one head $B = \{HH, HT, TH\}$

A and B are subsets of sample space S.

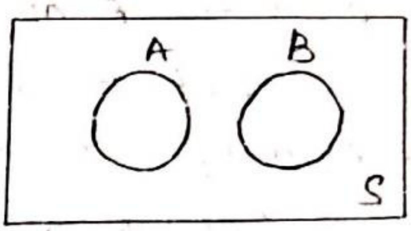
Trial and Event: Each performance of a random experiment is called a trial.

- Subset of sample space is called an event.

Types of events:

1. Mutually exclusive events
2. Favourable events
3. Independent events.

1. <u>Mutually</u> <u>Exclusive events</u>: Two sets A and B are said to be mutually exclusive or disjoint if they have no common elements i.e $A \cap B = \{\phi\}$



A = less than 5 = $\{1, 2, 3, 4\}$

B = greater than 5 = $\{6, 7, 8 \ldots \infty\}$.

$$A \cap B = \{\phi\}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = P(A) + P(B)$$

Q) A coin is tossed 4 times (i) Define the sample space 'S'.
(ii) Give the subsets and probabilities of the following events (a) more heads than tails
(b) tails occurs on even no. tosses.

Sol. Sample Space

(i) S = {HHHH, HHHT, HHTH, HTHH, THHH, HHTT, TTHH, THTH, HTHT, HTTT, THTT, TTTH, TTHT, TTTT, THHT, HTTH}.

(ii) more heads than tails = 5

A = {HHHH, HHHT, THHH, HHTH, HTHH}.

$$P(A) = \frac{5}{16}$$

tail occur on even no. tosses = 4

B = {HTHT, HTTT, TTHT, TTTT}

$$P(B) = \frac{4}{16} = \frac{1}{4}$$

* <u>Axiomatic defination of probability</u>: Let S be sample space and A be an event associated with a random experiment, then the probability of an event A denoted by P(A) is defined as a real number satisfying the following axioms.

(i) $0 \leq P(A) \leq 1$

(ii) $P(S) = 1$

(iii) If A and B are mutually exclusive events

$$P(A \cup B) = P(A) + P(B)$$

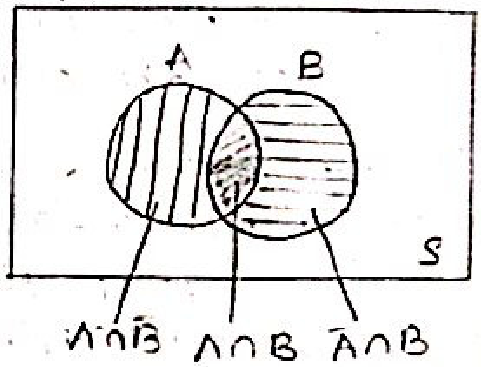
(iv) If $A_1, A_2 \dots A_n$ are mutually exclusive events then

$$P(A_1 \cup A_2 \cup A_3 \dots \cup A_n) = P(A_1) + P(A_2) + \dots$$

Theorem 1:

Addition theorem of two events: If A and B are two events then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof: $A \cap \bar{B}$ and $A \cap B$ are mutually exclusive events



$$A = (A \cap \bar{B}) \cup (A \cap B)$$

$$P(A) = P[(A \cap \bar{B}) \cup (A \cap B)]$$

$$P(A) = P(A \cap \bar{B}) + P(A \cap B) \text{ —— } ①$$

$$B = (A \cap B) \cup (\bar{A} \cap B)$$

$$P(B) = P[(A \cap B) \cup (\bar{A} \cap B)]$$

$$P(B) = P(A \cap B) + P(\bar{A} \cap B) \text{ —— } ②$$

[$\because A \cap B$ and $\bar{A} \cap B$ are mutually exclusive]

Adding ① and ②

$$P(A) + P(B) = P(A \cap \bar{B}) + P(A \cap B) + P(A \cap B) + P(\bar{A} \cap B)$$

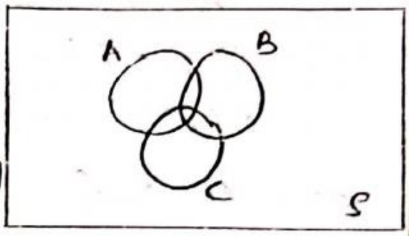
$$= P(A \cup B) + P(A \cap B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Theorem 2 :

Addition theorem of three events : If A, B, C are three events then $P(A\cup B\cup C) = P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$.

Proof : $P[A\cup(B\cup C)] = P(A)+P(B\cup C)-P[A\cap(B\cup C)]$



$= P(A)+P(B)+P(C)-P(B\cap C)-P[A\cap(B\cup C)]$

[According to addition theorem of 2 events]

$= P(A)+P(B)+P(C)-P(B\cap C)-\cancel{P}[P(A\cap B)\cup(A\cap C)]$

[Applying Distributive law]

$= P(A)+P(B)+P(C)-P(B\cap C)-[P(A\cap B)+P(A\cap C)-P(A\cap B)\cap(A\cap C)]$

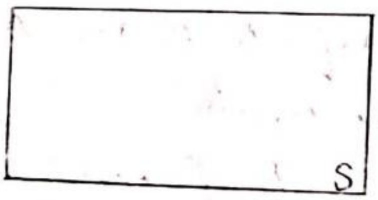
$= P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(A\cap C)\cancel{-P(B\cap C)}+P(A\cap(B\cap C))$.

$$P(A\cup B\cup C) = P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$$

Theorem 3 :

The probability of impossible events is zero i.e if ϕ is an event containing no sample point then $P(\phi)=0$.

Proof : S is the sample space and ϕ is the impossible event. S and ϕ are mutually exclusive events



$$P(S\cup\phi) = P(S)+P(\phi)$$

$$\because S\cup\phi = S$$

$$P(S\cup\phi) = P(S)$$

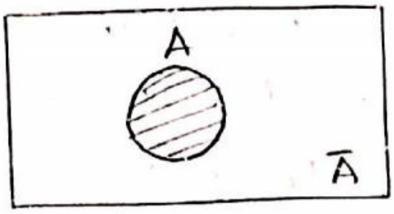
$$P(S) = P(S)+P(\phi)$$

$$P(\phi) = 0$$

Theorem 4:

If $\bar{A}$ is the complementary event of A then $P(\bar{A}) = 1 - P(A)$

Proof: A and $\bar{A}$ are mutually exclusive



$$A \cup \bar{A} = S$$

$$P(A \cup \bar{A}) = P(S)$$

$$P(A) + P(\bar{A}) = P(S)$$

Acc. to axiom (ii) of probability $P(S) = 1$

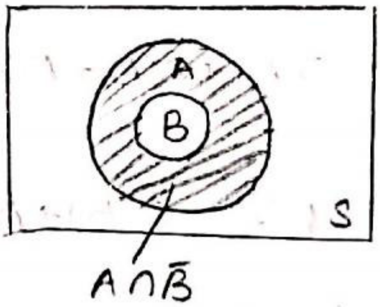
$$P(A) + P(\bar{A}) = 1$$

$$P(\bar{A}) = 1 - P(A)$$

Theorem 5:

If $B \subset A$, $P(B) \leq P(A)$

Proof: B and $A \cap \bar{B}$ are mutually exclusive



$$A = (A \cap \bar{B}) \cup B$$

$$P(A) = P[(A \cap \bar{B}) \cup B]$$

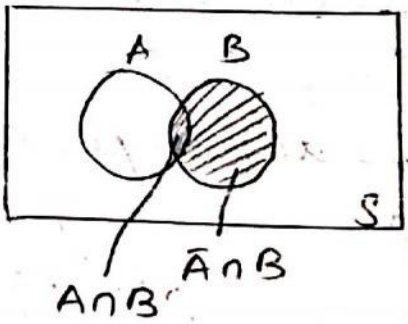
$$P(A) = P(A \cap \bar{B}) + P(B)$$

$$P(A) \geq P(B)$$

Theorem 6:

If A and B are two events then $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

Proof: $A \cap B$ and $\bar{A} \cap B$ are mutually exclusive



$$B = [(A \cap B) \cup (\bar{A} \cap B)]$$

$$P(B) = P[(A \cap B) \cup (\bar{A} \cap B)]$$

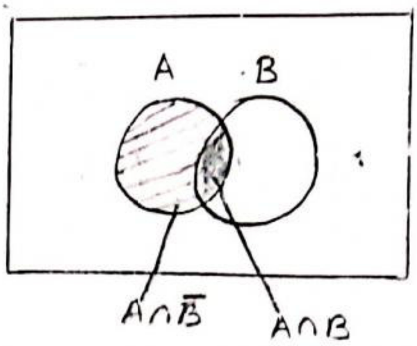
$$P(B) = P(A \cap B) + P(\bar{A} \cap B)$$

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

Theorem 7 : 1

(A) If A and B are two events then ~~$P(\bar{A}\cap\bar{B}) = P(B) - P(A$~~

$P(A\cap\bar{B}) = P(A) - P(A\cap B)$



Proof: $A\cap\bar{B}$ and $A\cap B$ are mutually exclusive

$$A = [(A\cap\bar{B})\cup(A\cap B)]$$

$$P(A) = P[(A\cap\bar{B})\cup(A\cap B)]$$

$$P(A) = P(A\cap\bar{B}) + P(A\cap B)$$

$$P(A\cap\bar{B}) = P(A) - P(A\cap B)$$

Q) If two dice are thrown, let A be the event of getting sum as 9 and B is the event of getting atleast one number as 6. Find the following probabilities

(i) $P(A\cap B)$ (ii) $P(A\cup B)$ (iii) $P(\bar{A}\cap B)$ (iv) $P(A\cap\bar{B})$ (v) $P(\bar{A}\cap\bar{B})$

(vi) $P(\bar{A}\cup\bar{B})$

Sol. A : getting sum as 9

S : two dice are rolled

$S = \{(1,1)(1,2)\ldots(6,6)\} = 36$

$A = \{(3,6)(6,3)(4,5)(5,4)\} = 4$

B : getting atleast one no. as 6

$B = \{(1,6)(2,6)(3,6)(4,6)(5,6)(6,6)(6,1)(6,2)(6,3)(6,4)$
$(6,5)\} = 11$

$A\cap B = \{(3,6)(6,3)\}$

$P(A) = \frac{4}{36}$ $\quad P(B) = \frac{11}{36}$ $\quad P(A\cap B) = \frac{2}{36}$

(i) $P(A\cap B) = \frac{2}{36}$

(ii) $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$$= \frac{4}{36} + \frac{11}{36} - \frac{2}{36}$$

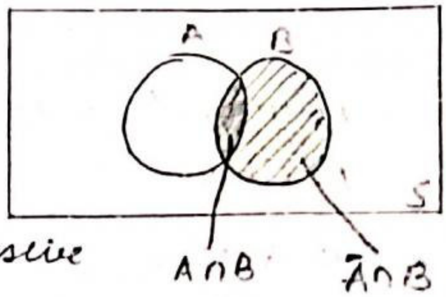
$$= \frac{13}{36}$$

$A_1, A_2, A_3 \ldots A_n$ are called totally independent

3m

Theorem 1:

If A and B are independent events then $\bar{A}$ and B are also independent



Proof: $B = (A \cap B) \cup (\bar{A} \cap B)$

$$P(B) = P[(A \cap B) \cup (\bar{A} \cap B)]$$

$A \cap B$ and $\bar{A} \cap B$ are mutually exclusive

$$P(B) = P(A \cap B) + P(\bar{A} \cap B)$$

Since A and B are independent events

$$P(A \cap B) = P(A) \times P(B)$$

$$P(B) = P(A) \times P(B) + P(\bar{A} \cap B)$$

$$P(B) - P(A) \times P(B) = P(\bar{A} \cap B)$$

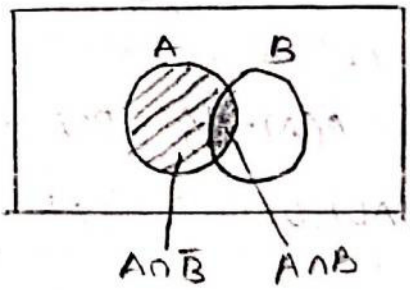
$$P(B)[1 - P(A)] = P(\bar{A} \cap B) \quad [\because P(\bar{A}) = 1 - P(A)]$$

$$P(\bar{A}) \times P(B) = P(\bar{A} \cap B)$$

$\therefore$ The events $\bar{A}$ and B are also independent events.

Theorem 2:

If A and B are independent events, then A and $\bar{B}$ are also independent.



Proof: $A = (A \cap \bar{B}) \cup (A \cap B)$

$$P(A) = P[(A \cap \bar{B}) \cup (A \cap B)]$$

Since $A \cap \bar{B}$ and $A \cap B$ are mutually exclusive

$$P(A) = P(A \cap \bar{B}) + P(A \cap B)$$

A and B are independent

$$P(A \cap B) = P(A) \times P(B)$$

$$P(A) = P(A \cap \bar{B}) + P(A) \times P(B)$$

$$P(A)[1 - P(B)] = P(A \cap \bar{B}) \quad [\because P(\bar{B}) = 1 - P(B)]$$

$$P(A) \times P(\bar{B}) = P(A \cap \bar{B})$$

$\therefore$ The events A and $\bar{B}$ are also independent

Theorem 3:

If A and B are independent then $\bar{A}$ and $\bar{B}$ are also independent events.

Proof: $P(\bar{A}\cap\bar{B}) = P(\overline{A\cup B})$

$= 1 - P(A\cup B)$ $\quad [\because P(\bar{A}) = 1 - P(A)]$

Applying addition theorem

$$P(\bar{A}\cap\bar{B}) = 1 - [P(A) + P(B) - P(A\cap B)]$$

Since A and B are independent

$$P(A\cap B) = P(A)\times P(B)$$

$$P(\bar{A}\cap\bar{B}) = 1 - [P(A) + P(B) - P(A)\times P(B)]$$

$$= 1 - P(A) - P(B) + P(A)\times P(B)$$

$$= 1 - P(A) - P(B)[1 - P(A)]$$

$$= [1 - P(A)][1 - P(B)]$$

$$= P(\bar{A})\,P(\bar{B})$$

$\bar{A}$ and $\bar{B}$ are independent events.

Q) The problem in statistics is given to 3 students, the probability of solving by the 1^{st} is $1/2$ and 2^{nd} is $3/4$ and 3^{rd} is $1/4$. What is the probability that the problem is solved.

Sol $P(A) = 1/2 \quad P(B) = 3/4 \quad P(C) = 1/4$

$$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(A\cap C) - P(B\cap C) + P(A\cap B\cap C)$$

$$= P(A) + P(B) + P(C) - P(A)\times P(B) - P(A)\times P(C) - P(B)\times P(C) + P(A)\times P(B)\times P(C)$$

$[\because A, B, C$ are independent events$]$

$$= \frac{1}{2} + \frac{3}{4} + \frac{1}{4} - \frac{1}{2}\times\frac{3}{4} - \frac{1}{2}\times\frac{1}{4} - \frac{3}{4}\times\frac{1}{4} + \frac{1}{2}\times\frac{3}{4}\times\frac{1}{4}$$

$$= \frac{1}{2} + \frac{3}{4} + \frac{1}{4} - \frac{3}{8} - \frac{1}{8} - \frac{3}{16} + \frac{3}{32} = \frac{29}{32}$$

$$= 0.90625$$

(OR)

$$P(A \cup B \cup C) = 1 - P(\overline{A \cup B \cup C})$$
$$= 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$$
$$= 1 - P(\bar{A})\, P(\bar{B})\, P(\bar{C})$$
$$= 1 - \frac{1}{2} \times \frac{1}{4} \times \frac{3}{4}$$
$$= 1 - \frac{3}{32}$$
$$= \frac{29}{32}$$

Q) Two fair dice are thrown independent. Event A is defined as odd face with the first die. Event B is odd face with the second die. Event C is sum of numbers in two dice is odd. Are the events A, B, C mutually independent? Are the event pairwise independent.

Sol. Sample space = 36

$P(A) = \{(1,1)\ (3,1)\ (5,1)$
$(1,2)\ (3,2)\ (5,2)$
$(1,3)\ (3,3)\ (5,3)$
$(1,4)\ (3,4)\ (5,4)$
$(1,5)\ (3,5)\ (5,5)$
$(1,6)\ (3,6)\ (5,6)\}$

$B = \{(1,1)\ (1,3)\ (1,5)$
$(2,1)\ (2,3)\ (2,5)$
$(3,1)\ (3,3)\ (3,5)$
$(4,1)\ (4,3)\ (4,5)$
$(5,1)\ (5,3)\ (5,5)$
$(6,1)\ (6,3)\ (6,5)\}$

$$P(A) = \frac{18}{36} = \frac{1}{2}$$

P(B) is also same

$$P(B) = \frac{18}{36} = \frac{1}{2}$$

$P(C)$ = sum odd
= odd + even
$= \{(1,2)\ (1,4)\ (1,6)\ (3,2)\ (3,4)\ (3,6)\ (5,2)\ (5,4),$
$(5,6)\ (2,1)\ (4,1)\ (6,1)\ (2,3)\ (4,3)\ (6,3),\ (2,5)$
$(4,5)\ (6,5)\}$

$$P(C) = \frac{18}{36} = \frac{1}{2}$$

$P(A\cap B) = \{(1,1)(1,3)(1,5)(5,1)(3,5)(5,3)(5,5)(3,3)(3,1)\}$

$= \frac{9}{36} = \frac{1}{4}$

$B\cap C = \{(2,3)(2,1)(4,1)(6,1)(4,3)(6,3)(4,5)(6,5)(2,5)\}$

$P(B\cap C) = \frac{9}{36} = \frac{1}{4}$

$A\cap C = \{(1,2)(3,2)(1,4)(5,6)(1,6)(5,2)(3,4)(3,6)(5,4)\}$

$P(A\cap C) = \frac{9}{36} = \frac{1}{4}$

$P(A\cap B) = \frac{1}{4}$

$P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

A and B are pairwise independent

$P(B\cap C) = \frac{1}{4}$

$P(B) \times P(C) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$\therefore$ B and C are pairwise independent

$P(A\cap C) = \frac{1}{4}$

$P(A) \times P(C) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$\therefore$ A and C are pairwise independent.

$(A\cap B\cap C) = \{0\}$

$P(A\cap B\cap C) = 0$

$P(A) \times P(B) \times P(C) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

A and B and C are not mutually independent.

A can win the game in first round or in the second round (i.e. in first round both A & B lost and in second A wins) similarly A can win in the third round (