$$= \left[\frac{x^3 e^{-x}}{-1} \right]_0^\infty - \left[\frac{3x^2 e^{-x}}{-1} + \int 6x\, e^{-x} \right]_0^\infty$$

$$= \left[-x^3 e^{-x} \right]_0^\infty - \left[\frac{3x^2 e^{-x}}{-1} + \frac{6x e^{-x}}{-1} - 6x^{-x} \right]_0^\infty$$

$$= [0-0] + [0+0-6(0-1)]$$

$$= 6$$

$$Var(X) = 6-(2)^2$$
$$= 2$$

Special Random Variable:

Discrete

1) Binomial Distribution
2) Poisson Distribution
3) Geometric Distribution

Continuous

1) Uniform or Rectangular Distribution
2) Exponential Distribution
3) Erlang or Gamma Distribution
4) Normal or Gaussian Distribution

Binomial Distribution: Let A be some event associated with a random expt E, such that

$$P(A) = p$$
$$P(\bar{A}) = 1-p$$
$$= q$$

- Assuming that p remains same for all repetations, ~~for all~~ if we consider n independent repetations of E and if random variable X denotes the no. of times event A has occured then X is called Binomial Random Variable ~~x~~ with parameters n and p or we say that X follows Binomial Distribution

$$\sum_{x=0}^{\infty} P(X=x) = {}^nC_x\, p^x (q)^{n-x}$$
$$= (p+q)^n = 1$$

By the theorem, under Bernoulli's trials, the probability mass function of binomial random variable is given as $P(X=r) = {}^nC_r\, p^r q^{n-r}$ $\quad r = 0, 1, 2 \ldots n$

where $p+q=1$

Mean of Binomial Distribution:

$$\text{Mean} = E(X) = \sum_{r=0}^{n} r\,P(r) \quad \Sigma \wedge P(X=r)$$

$$= \sum_{r=0}^{n} r\, {}^nC_r\, p^r q^{n-r}$$

$$= \sum_{r=0}^{n} r \cdot \frac{n!}{(n-r)!\,r!}\, p^r q^{n-r}$$

$$= \sum_{r=0}^{n} \frac{n!}{(n-r)!\,(r-1)!}\, p^r q^{n-r}$$

$$= \sum_{r=1}^{n} \frac{(n-1)!}{(n-r)!\,(r-1)!}\, p^r q^{n-r}$$

$$= n \sum_{r=1}^{n} \frac{(n-1)!}{(n-r)!\,(r-1)!}\, p^r q^{n-r}$$

Replace $n-r$ with $[(n-1)-(r-1)]$

$$= n \sum_{r=1}^{n} \frac{(n-1)!}{[(n-1)-(r-1)]!\,(r-1)!}\, p^r q^{[(n-1)-(r-1)]}$$

$$= n \sum_{r=1}^{n} {}^{(n-1)}C_{r-1}\, p^r q^{(n-1)-(r-1)}$$

Multiply and divide by p.

$$= np \sum_{r=1}^{n} {}^{(n-1)}C_{r-1}\, p^{r-1} q^{(n-1)-(r-1)}$$

$$= np(p+q)^{n-1}$$

$$E(X) = np.$$

Variance of Binomial Distribution:

$$Var(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \sum_{r=0}^{n} r^2 p_r.$$

$$E(X^2) = \sum_{r=0}^{n} r^2 \, nC_r \, p^r q^{n-r}$$

$$r^2 = r(r-1) + r$$

$$E(X^2) = \sum_{r=0}^{n} [r(r-1) + r] \, nC_r \, p^r q^{n-r}$$

$$= \sum_{r=0}^{n} r(r-1) \, nC_r \, p^r q^{n-r} + \sum_{r=0}^{n} r \, nC_r \, p^r q^{n-r}$$

$$= \sum_{r=0}^{n} r(r-1) \frac{n!}{(n-r)!\, r!} p^r q^{n-r} + \sum_{r=0}^{n} r p_r.$$

$$= \sum_{r=2}^{n} \frac{n!}{(n-r)!\,(r-2)!} p^r q^{n-r} + np.$$

$$= n(n-1) \sum_{r=2}^{n} \frac{(n-2)!}{(n-r)!\,(r-2)!} p^r q^{n-r} + np$$

Replace $n-r$ with $[(n-2)-(r-2)]$

$$E(X^2) = n(n-1) \sum_{r=2}^{n} \frac{(n-2)!}{[(n-2)-(r-2)]!\,(r-2)!} p^r q^{[(n-2)-(r-2)]} + np$$

$$= n(n-1) \sum_{r=2}^{n} (n-2)C_{r-2} \, p^r q^{(n-2)-(r-2)} + np$$

Multiply and divide with p^2

$$= p^2 n(n-1) \left[\sum_{r=2}^{n} (n-2)C_{r-2} \, p^{r-2} q^{(n-2)-(r-2)}\right] + np.$$

$$E(X^2) = p^2 n(n-1) (p+q)^{n-2} + np.$$

$$\sum_{r=0}^{n} nC_r \, p^r q^{n-r} = (p+q)^n.$$

$$E(X^2) = p^2(n^2 - n) + np.$$

$$= n^2p^2 - np^2 + np.$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np(1-p)$$

$$= npq$$

7) The mean and variance of binomial distribution are 4 and 4/3 respectively. Find $P(X \geq 1)$

sol: $\mu = np = 4$

$var(x) = npq = \frac{4}{3}$

$4q = \frac{4}{3}$

$q = \frac{1}{3}$

$p = \frac{2}{3}$

$npq = \frac{4}{3}$

$n\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \frac{4}{3}$

$n = 6$

$P(X \geq 1) = \sum_{r=1}^{6} nC_r \, p^r q^{n-r} = 1 - P(X<1)$

$P(X \geq 1) = 1 - P(X=0)$

$= 1 - 6C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^{6-0}$

$= 1 - \left(\frac{1}{3}\right)^6$

$= 1 - \frac{1}{729}$

$= 0.998$

8) Two dice are thrown 120 times. Find the average no. of times in which the no. on the first dice exceeds the no. on the second dice.

sol: $n = 120$

Mean $= np$

Success $= \{(2,1), (3,1) (3,2) (4,1) (4,2) (4,3) (5,1) (5,2) (5,3) (5,4) (6,1) (6,2) (6,3), (6,4) (6,5)\}$

$p = \frac{15}{36} = \frac{5}{12}$

$\mu = 120 \times \frac{5}{12} = 50$

<u>Poisson Distribution</u> : If X is a discrete random variable, that can assume the values $0, 1, 2 \ldots \infty$ such that the probability mass function is given by

$$P(X=r) = \frac{e^{-\lambda} \cdot \lambda^{r}}{r!} \; ; \; r = 0, 1, 2 \ldots \quad \lambda > 0$$

then X is said to follow Poisson's distribution with the parameter λ denoted by $P(\lambda)$ 'λ' $\quad \lambda = np = \text{mean}$

<u>Poissons Distribution as limiting form of Binomial Distribution</u> : Poisson distribution is the limiting case of binomial distribution under following conditions

1) The no of trials 'n' is indefinitely large.

$$n \to \infty$$

2) Probability of success 'p' in each trial is very small

$$p \to 0$$

3) $np = \lambda$

$$p = \frac{\lambda}{n} \qquad q = 1 - \frac{\lambda}{n}$$

np is finite and λ is a positive real no.

<u>Proof</u>: If X is binomially distributed random variable with parameters n, p then.

$$P(X=r) = nC_r \, p^r q^{n-r}$$

$$= \frac{n!}{(n-r)!\, r!} \left(\frac{\lambda}{n}\right)^r \left(1 - \frac{\lambda}{n}\right)^{n-r}$$

$$= \frac{n(n-1)(n-2)\ldots[(n-(r-1)]}{r!} \left(\frac{\lambda}{n}\right)^r \left(1 - \frac{\lambda}{n}\right)^{n-r}$$

$$= \frac{n \cdot n^{r-1}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{(r-1)}{n}\right)}{r!} \frac{\lambda^r}{n^r} \times \left(1 - \frac{\lambda}{n}\right)^{n-r}$$

$$= \frac{\lambda^r}{r!} \lim_{n\to\infty} \left[\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\left(\frac{r-1}{n}\right)\right)\right] \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{n-r}$$

$$= \frac{\lambda^r}{r!} (1) \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n \left[\lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{-r}\right]$$

$$= \frac{\lambda^r}{r!} (1) \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n (1)$$

$$= \frac{\lambda^r}{r!} e^{-\lambda}$$

$$\left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$$

$$\left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$$

$$= \frac{e^{-\lambda} \lambda^r}{x!}$$

<u>Mean</u> <u>of</u> <u>Poisson</u> <u>Distribution</u>:

$$E(X) = \sum X_r P_r$$

$$= \sum_{r=0}^{\infty} r \left(\frac{e^{-\lambda} \lambda^r}{r!}\right)$$

$$E(X) = \sum_{r=1}^{\infty} \frac{e^{-\lambda} \lambda^r}{(r-1)!}$$

$$E(X) = e^{-\lambda} \lambda^1 \sum_{r=0}^{\infty} \frac{\lambda^{r-1}}{(r-1)!}$$

$$= e^{-\lambda} \lambda^1 \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots \infty\right]$$

$$\cancel{= e^{-\lambda} \sum \lambda (e^{\infty}}$$

$$= e^{-\lambda} \lambda e^{\lambda})$$

$$= \lambda$$

<u>Variance</u> <u>of</u> <u>Poisson</u> <u>Distribution</u>

$$Var(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \sum r^2 P_r$$

$$= \sum_{r=0}^{\infty} r^2 \left(\frac{e^{-\lambda} \lambda^r}{r!}\right)$$

Replace $r^2 = r(r-1) + r$

$$E(X^2) = \sum_{r=0}^{\infty} [r(r-1) + r] \frac{e^{-\lambda} \lambda^r}{r!}$$

$$= \sum_{r=0}^{\infty} \left(\frac{r(r-1) e^{-\lambda} \lambda^r}{r!}\right) + \sum_{r=0}^{\infty} r \left(\frac{e^{-\lambda} \lambda^r}{r!}\right)$$

$$E(X^2) = \sum_{r=2}^{\infty} \frac{e^{-\lambda}\lambda^r}{(r-2)!} + E(X)$$

$$= e^{-\lambda}\lambda^2 \sum_{r=0}^{\infty} \frac{\lambda^{r-2}}{(r-2)!} + \lambda$$

$$= e^{-\lambda}\lambda^2 \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \dots \infty\right] + \lambda$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = e^{-\lambda}\lambda^2(e^{\lambda}) + \lambda$$

$$E(X^2) = \lambda^2 + \lambda$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$= \lambda$$

Q) The no. of ~~mott~~ monthly breakdowns of a computer is a random variable having poisson's distribution with mean = 1.8. Find the probability that.

(i) computer will function for a month without breakdown

(ii) with one breakdown

(iii) atleast one breakdown

Sol: Given $\lambda = 1.8$

(i) $r = 0$; $\lambda = 1.8$

$$P(X=0) = \frac{e^{-\lambda}\lambda^r}{r!}$$

$$= \frac{e^{-1.8}(1.8)^0}{0!}$$

$$= e^{-1.8}$$

$$= 0.1653$$

(ii) $r = 1$

$$P(r=1) = \frac{e^{-\lambda}\lambda^r}{r!}$$

$$= \frac{e^{-1.8}(1.8)^1}{1!}$$

$$= e^{-1.8}(1.8)^1$$

$$= 0.297$$

(iii) Atleast one breakdown

$$P(X=r) = 1 - P(r=0)$$
$$= 1 - \frac{e^{-\lambda}\lambda^{r}}{r!}$$
$$= 1 - \frac{e^{-1.8}(1.8)^0}{0!}$$
$$= 1 - e^{-1.8}$$
$$= 1 - 0.1653$$
$$= 0.8343$$

2) A manufacturer of pins knows that size (5%) of his pins are defective. If he sells pins in the boxes of 100 and guarantee is that not more than 10 pins will be defective. What is the approx. probability that a box will fail to meet the guaranteed quality.

Sol: $P(X>10) = 1 - P(X \leq 10)$

$p = 0.05 \quad q = 0.95$

$n = 100 \quad \lambda = np$

$= 100 \times 0.05$

$\lambda = 5$

$$P(X>10) = 1 - \sum_{r=0}^{10} \frac{e^{-5}(5)^r}{r!}$$

Geometric Distribution : If x is a discrete random variable that can assume values $1, 2, 3 \ldots \infty$, then, the probability mass function is given by $P(X=r) = q^{r-1}p^1$ where $p+q=1$ then X is said to follow geometric distribution

$$\sum_{r=1}^{\infty} P(X=r) = \sum_{r=1}^{\infty} q^{r-1}p^1$$
$$= p[q^0 + q^1 + q^2 + \ldots \infty]$$
$$S = \frac{a}{1-r} = p\left[\frac{1}{1-q}\right]$$
$$= p\left[\frac{1}{p}\right] = 1$$

$$\sum_{r=1}^{\infty} P(X=r) = 1$$

Mean $\mu = \sum x p_x$

$$= \sum_{x=1}^{\infty} x q^{x-1} p^1$$

$$= p \sum_{x=1}^{\infty} q^{x-1}$$

$$= p[1 + 2q^1 + 3q^2 + 4q^3 + \dots \infty]$$

$$= p(1-q)^{-2}$$

$$= p \cdot p^{-2}$$

$$= \frac{p}{p^2} = \frac{1}{p}$$

$$\mu = \frac{1}{p}$$

varience : $Var(x) = E(x^2) - [E(x)]^2$

$$E(x^2) = \sum x^2 p_x$$

$$= \sum_{x=1}^{\infty} x^2 q^{x-1} p^1$$

Replace $x^2 = x(x-1) + x$

$$E(x^2) = \sum_{x=1}^{\infty} [x(x-1) + x] q^{x-1} p$$

$$= \sum_{x=1}^{\infty} x(x-1) q^{x-1} p + \underbrace{\sum_{x=1}^{\infty} x q^{x-1} p^1}_{\text{mean}}$$

$$= p \cdot \sum_{x=1}^{\infty} x(x-1) q^{x-1} + \frac{1}{p}$$

$$= p[0 + 2 \cdot 1 \cdot q^1 + 3 \cdot 2 q^2 + 4 \times 3 q^3 + \dots \infty] + \frac{1}{p}$$

$$= p \cdot (2q)[1 + 3q + 6q^2 + \dots \infty] + \frac{1}{p}$$

$$= p(2q)(1-q)^{-3} + \frac{1}{p}$$

$$= p(2q)(p)^{-3} + \frac{1}{p}$$

$$= 2qp^{-2} + \frac{1}{p}$$

$$= \frac{2(1-p)}{p^2} + \frac{1}{p}$$

$$= \frac{2}{p^2} - \frac{2}{p} + \frac{1}{p}$$

$$= \frac{2}{p^2} - \frac{1}{p}$$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} = \frac{1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2} = \frac{q}{p^2}$$

Q) If the probability that an applicant for driver's licence will pass the road test on any trial is 0.8. What is the probability that he will pass the test on 4^{th} trial.
(ii) less than 4 trials.

sol $p=0.8 \quad q=0.2 \cdot \quad x=4$

(i) $P(X=4) = q^{x-1} p^1$

$= (0.2)^3 (0.8)^1$

$= 0.0064$

(ii) $P(X<4) = P(X=1) + P(X=2) + P(X=3)$

$= q^0 p^1 + q^1 p^1 + q^2 p^1$

$= 0.8 [1 + (0.2) + (0.2)^2]$

$= 0.992.$

Continuous type Random Variables:

1) Uniform or Rectangular Distribution: A continuous type random variable 'x' is said to follow uniform distribution in any finite interval. ~~If its~~

- If its probability density function is a constant in that interval.
- If 'x' follows a uniform distribution in the interval $a<x<b$, then $f(x) = \frac{1}{b-a}$,
- When x follows a uniform distribution in the interval (a,b), let $f(x) = k$.

$\int f(x)\,dx = 1$

$$\int_a^b k\,dx = 1$$

$$k(x)_a^b = 1$$

$$k(b-a) = 1$$

$$k = \frac{1}{b-a}$$

$$f(x) = \frac{1}{b-a}, \quad a<x<b$$

Uniform distribution is denoted by $U(a,b)$

Mean: $\int_a^b x f(x)\,dx = \int_a^b \frac{x}{b-a}\,dx$

$$= \frac{1}{b-a}\left(\frac{x^2}{2}\right)_a^b$$

$$= \frac{b^2-a^2}{2(b-a)}$$

$$= \frac{a+b}{2}$$

<u>Varience</u> : $E(X^2) - [E(X)]^2$

$$E(X^2) = \frac{1}{b-a}\int_a^b x^2 dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{1}{b-a}(b^3-a^3)$$

$$= \frac{1}{b-a}\frac{(b-a)(a^2+b^2+ab)}{3}$$

$$= \frac{a^2+b^2+ab}{3}$$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= \frac{a^2+b^2+ab}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{a^2+b^2+ab}{3} - \left(\frac{a^2+b^2+2ab}{4}\right)$$

$$= \frac{4a^2+4b^2+4ab-3a^2-3b^2-6ab}{12} = \frac{a^2+b^2-2ab}{12}$$

$$= \frac{(a-b)^2}{12}$$

Q) Find the mean and varience of a random variable uniformly distributed in the interval (0, 10) !

<u>sol</u> $\mu = \frac{a+b}{2} = \frac{0+10}{2}$

$\mu = 5$

$$\sigma^2 = var = \left(\frac{a-b}{12}\right)^2$$

$$= \left\{\frac{(0-10)}{12}\right\}^2$$

$$= \frac{25}{3}$$

<u>Exponential</u> <u>Distribution</u> : A continuous random variable X is said to follow an exponential distribution or ~~rana~~ negative exponential distribution with parameters $\lambda > 0$ if its probability density function is given by :

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & ; x \geq 0 \\ 0 & ; \text{elsewhere} \end{cases}$$

Note that,

$$\int_0^{\infty} f(x)\,dx = \int_0^{\infty} \cancel{x}\,\lambda e^{-\lambda x}\,dx = 1$$

$$= \text{ll}$$

- If this is true, then $f(x)$ is a legitimate density function.

Mean of Exponential Distribution:

$$\mu = \int_0^{\infty} x f(x)\,dx$$

$$= \int_0^{\infty} x \lambda e^{-\lambda x} dx.$$

$$= \lambda \int_0^{\infty} \underset{\mathrm{I}}{x}\, \underset{\mathrm{II}}{e^{-\lambda x}}\,dx$$

$$= \lambda \left[\frac{x e^{-\lambda x}}{-\lambda} - \int \frac{1 \cdot e^{-\lambda x}}{-\lambda} dx \right]_0^{\infty}$$

$$= \lambda \left[\frac{-x}{\lambda} e^{-\lambda x} - \frac{e^{-\lambda x}}{\lambda^2} \right]_0^{\infty}$$

$$= \lambda \left[\frac{-\infty \times 0 + 0}{\lambda} - \frac{(e^{-\infty} - e^{-0})}{\lambda^2} \right]$$

$$= \lambda \times \frac{1}{\lambda^2}.$$

$$= \frac{1}{\lambda}$$

Varience of Exponential distribution:

$$Var(X) = E(X^2) - [E(X)]^2$$

$$\cancel{E(X^2) = \int_0^{\infty} x^2 \cdot \lambda e^{-\lambda x} dx.}$$

$$E(X^2) = \int_0^{\infty} x^2 f(x)\,dx.$$

$$= \int_0^{\infty} x^2 \cdot \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^{\infty} \underset{\mathrm{I}}{x^2}\, \underset{\mathrm{II}}{e^{-\lambda x}}\,dx$$

$$= \lambda \left[\frac{x^2 e^{-\lambda x}}{-\lambda} - \int \frac{2x\, e^{-\lambda x}}{-\lambda} \right]_0^\infty$$

$$= \lambda \left[\frac{-x^2 e^{-\lambda x}}{\lambda} + \frac{2x\, e^{-\lambda x}}{\lambda^2} - \int \frac{2 e^{-\lambda x}}{-\lambda^2} \right]_0^\infty$$

$$= \lambda \left[\frac{-x^2 e^{-\lambda x}}{\lambda} + \frac{2x e^{-\lambda x}}{\lambda^2} - \frac{2e^{-\lambda x}}{\lambda^3} \right]_0^\infty$$

$$= \lambda \left[\frac{-\infty \times 0 + 0}{\lambda} + \frac{\infty \times 0 - 0}{\lambda^2} - \frac{2}{\lambda^3}(e^{-\infty} - e^0) \right]$$

$$= \lambda \left[\frac{2}{\lambda^3} \right]$$

$$E(x^2) = \frac{2}{\lambda^2}$$

$$var(x) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{1}{\lambda^2}$$

& $\Gamma k = (k-1)!$

$$\int_0^\infty e^{-t}\, t^{k-1}\, dt = \Gamma k$$

$$\Gamma k+1 = k\,\Gamma k$$

$$\Gamma k+2 = \Gamma (k+1)+1 = (k+1)\,\Gamma k+1$$

~~Erlang or Gamma function: A continuous~~

<u>Erlang</u> <u>or</u> <u>Gamma</u> <u>distribution</u>: A continuous random variable 'X' is said to follow Gamma or Erlang distribution with parameters $\lambda > 0$ and $k > 0$ if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{\lambda^k e^{-\lambda x}\, x^{k-1}}{\Gamma k}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$\int_0^\infty f(x)\, dx = \int_0^\infty \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma k} dx$$

$$= \frac{1}{\Gamma k} \int_0^\infty \frac{(\lambda x)^k}{x} e^{-\lambda x} dx$$

$$\lambda x = t$$
$$\lambda dx = dt$$

$$\int_0^\infty f(x)\,dx = \frac{1}{\Gamma k}\int_0^\infty \frac{(t)^k e^{-t}}{(t/\lambda)}\,\frac{dt}{\lambda}$$
$$= \frac{1}{\Gamma k}\int_0^\infty e^{-t}\,t^{k-1}\,dt = \frac{1}{\Gamma k}\,\Gamma k = 1.$$
$$\therefore \int_0^\infty e^{-t}\,t^{x-1}\,dt = \Gamma x$$

$\therefore$ f(x) is a legitimate density function.

<u>Mean</u> $\mu = \int_0^\infty x \cdot f(x)\,dx.$

$$= \int_0^\infty \frac{x\,\lambda^k e^{-\lambda x}\,x^{k-1}}{\Gamma k}\,dx$$
$$= \frac{1}{\Gamma k}\int_0^\infty \frac{\lambda^k e^{-\lambda x}\,x^k}{\cancel{\Gamma k}}\,dx.$$
$$= \frac{1}{\Gamma k}\int_0^\infty (\lambda x)^k e^{-\lambda x}\,dx.$$

Put $\lambda x = t$

$\lambda\,dx = dt$

$$\mu = \frac{1}{\Gamma k}\int_0^\infty t^k e^{-t}\,\frac{dt}{\lambda}$$
$$= \frac{1}{\Gamma k\,\lambda}\int_0^\infty t^{(k+1)-1} e^{-t}\,dt.$$

We know

$$\int_0^\infty e^{-t}\,t^{k-1}\,dt = \Gamma k.$$
$$\int_0^\infty e^{-t}\,t^{(k+1)-1}\,dt = \Gamma(k+1).$$
$$\mu = \frac{1}{\Gamma k\,\lambda}\,\Gamma(k+1).$$
$$= \frac{1}{\lambda\,\Gamma k}\,k\,\Gamma k = \frac{k}{\lambda}$$

Variance $= E(X^2) - [E(X)]^2$

$$E(X^2) = \int_0^\infty x^2 \frac{x^{k-1}\lambda^k e^{-\lambda x}}{\Gamma k}$$

$$= \frac{1}{\Gamma k}\int_0^\infty \lambda^k x^{k+1} e^{-\lambda x}\,dx.$$

$$= \frac{1}{\Gamma k}\int_0^\infty x^k (x\lambda)^k x\, e^{-\lambda x}\,dx$$

Put $\lambda x = t$.

$\lambda dx = dt$

$$= \frac{1}{\Gamma k}\int_0^\infty t^k e^{-t} \frac{t}{\lambda}\frac{dt}{\lambda}$$

$$\cancel{= \frac{1}{\lambda^2}\frac{1}{\Gamma k}\int_0^\infty t^{k+1} e^{-t} dt}$$

$$\cancel{= \frac{1}{\lambda \Gamma k}\int_0^\infty}$$

$$= \frac{1}{\lambda^2 \Gamma k}\int_0^\infty t^{k+1} e^{-t}\,dt$$

$$= \frac{1}{\lambda^2 \Gamma k}\int_0^\infty t^{(k+2)-1} e^{-t}\,dt.$$

$$\int_0^\infty e^{-t} t^{k-1}\,dt = \Gamma k.$$

$$\int_0^\infty e^{-t} t^{(k+2)-1}\,dt = \Gamma k+2$$

$$E(X^2) = \frac{1}{\lambda^2 \Gamma k}\Gamma k+2$$

$$= \frac{1}{\lambda^2 \Gamma k}\Gamma (k+1)+1$$

$$= \frac{1}{\lambda^2 \Gamma k}(k+1)\Gamma k+1$$

$$= \frac{1}{\lambda^2 \Gamma k}(k+1)k\Gamma k.$$

$$= \frac{k(k+1)}{\lambda^2}$$

Variance : $E(X^2) - [E(X)]^2$

$$= \frac{k(k+1)}{\lambda^2} - \frac{k^2}{\lambda^2}$$

$$= \frac{k^2}{\lambda^2} + \frac{k}{\lambda^2} - \frac{k^2}{\lambda^2}$$

$$= \frac{k}{\lambda^2}$$

Normal or Gaussian Distribution: A continuous random variable is said to follow normal or gaussian distribution with parameters μ and σ if its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} & ; -\infty < x < \infty \\ 0 & ; \text{otherwise} \end{cases}$$

denoted by $N(\mu, \sigma)$ or $N(\mu, \sigma^2)$

Sol $\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx$

Put $\frac{x-\mu}{\sqrt{2}\sigma} = t$

$\frac{dx}{\sqrt{2}\sigma} = dt$

$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t}(\sqrt{2}\sigma\, dt)$

$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt$

$f(-t) = e^{-(-t^2)}$
$= e^{-t^2}$.
$f(t) = e^{-(+t^2)}$
$= e^{-t^2}$

$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 0$; if $f(x)$ is odd.

$\int_{-\infty}^{\infty} f(x)\,dx = 2\int_{0}^{\infty} f(x)\,dx$, if $f(x)$ is even function

Put $t^2 = z$.
$2t\,dt = dz$.

$= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-z} \frac{dz}{2\sqrt{z}}$

$= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-z} z^{-1/2} dz$

$= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-z} z^{1/2-1} dz$

$\because \int_{0}^{\infty} e^{-t} t^{k-1} dt = \Gamma k$.

$= \frac{1}{\sqrt{\pi}} \Gamma\frac{1}{2}$

$= \frac{1}{\sqrt{\pi}} \sqrt{\pi}$

$= 1$

$\therefore f(x)$ is a legitimate density function.

Mean of Normal Distribution :

$$\mu = \int_{-\infty}^{\infty} x \cdot f(x)\, dx .$$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx .$$

Put $\frac{x-\mu}{\sqrt{2}\sigma} = t$ $\qquad \sqrt{2}\sigma t + \mu = x$

$$\frac{dx}{\sqrt{2}\sigma} = dt$$

$$\text{Mean} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} (\sqrt{2}\sigma t + \mu) \sqrt{2}\sigma\, dt$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \sqrt{2}\sigma t e^{-t^2} dt + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mu e^{-t^2}$$

$$= \frac{\sqrt{2}\sigma}{\sqrt{\pi}} \int_{-\infty}^{\infty} t e^{-t^2} dt + \frac{\mu}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt .$$

$t e^{-t^2}$ - odd
e^{-t^2} - even

$$= 0 + \frac{\mu}{\sqrt{\pi}} \int_{0}^{\infty} 2 e^{-t^2} dt$$

Put $t^2 = z$

$2t\, dt = dz .$

$$\text{Mean} = \frac{2\mu}{\sqrt{\pi}} \int_{0}^{\infty} e^{-z} \frac{dz}{2\sqrt{z}}$$

$$= \frac{\mu}{\sqrt{\pi}} \int_{0}^{\infty} e^{-z} z^{\frac{1}{2}-1} dz$$

$$= \frac{\mu}{\sqrt{\pi}} \Gamma\frac{1}{2} = \mu \frac{\sqrt{\pi}}{\sqrt{\pi}}$$

$$= \mu .$$

Variance :

$$\text{Variance} = E(x^2) - [E(x)]^2$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 \times \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} dx .$$

put $\frac{x-\mu}{\sqrt{2}\sigma} = t ,$

$x = \sqrt{2}\sigma t + \mu .$

$$\frac{dx}{\sqrt{2}\sigma} = dt$$

$$E(x^2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} (\sqrt{2}\sigma t + \mu)^2 \sqrt{2}\sigma\, dt$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (2\sigma^2 t^2 + \mu^2 + 2\sqrt{2}\sigma t \mu) e^{-t^2} dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2} dt + \frac{\mu^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt + \frac{2\sqrt{2}\sigma\mu}{\sqrt{\pi}} \int_{-\infty}^{\infty} t e^{-t^2} dt$$

$t^2 e^{-t^2}$ is even function

e^{-t^2} is even function

te^{-t^2} is odd function

$$E(x^2) = \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} 2t^2 e^{-t^2} dt + \frac{\mu^2}{\sqrt{\pi}} \int_0^{\infty} 2e^{-t^2} dt$$

put $t^2 = z$

$2t\,dt = dz$

$$E(x^2) = \frac{4\sigma^2}{\sqrt{\pi}} \int_0^{\infty} ze^{-z} \frac{dz}{2\sqrt{z}} + \frac{2\mu^2}{\sqrt{\pi}} \int_0^{\infty} e^{-z} \frac{dz}{2\sqrt{z}}$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} e^{-z} z^{1/2} dz + \frac{\mu^2}{\sqrt{\pi}} \int_0^{\infty} e^{-z} z^{-1/2} dz.$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} e^{-z} e^{3/2-1} dz + \frac{\mu^2}{\sqrt{\pi}} \int_0^{\infty} e^{-z} z^{1/2-1} dz.$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\tfrac{3}{2} + \frac{\mu^2}{\sqrt{\pi}} \Gamma\tfrac{1}{2}$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\tfrac{1}{2}+1\right) + \frac{\mu^2}{\sqrt{\pi}} \sqrt{\pi}$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2} \Gamma\tfrac{1}{2} + \mu^2$$

$$= \frac{\sigma^2}{\sqrt{\pi}} \sqrt{\pi} + \mu^2$$

$$Var(X) = E(x^2) - E(x^2)$$

$$= \sigma^2 + \mu^2 - \mu^2$$

$$= \sigma^2$$

Asymptotic Approximations of Binomial Distribution

Let X represent Binomial random variable

$P(X=r) = {}^nC_r\, p^r q^{n-r}$ then

$$P(r_1 \le x \le r_2) = \sum_{r=r_1}^{r_2} {}^nC_r\, p^r q^{n-r}.$$

since the binomial coefficient ${}^nC_r = \frac{n!}{(n-r)!\,r!}$

grows quite rapidly with n, it is difficult to compute eqn(1) for very large n.

There are 2 approximations used when n becomes very large

1) Poisson approximation
2) Normal approximation/De-Moivre Laplace theorem

Poissons approximation:

Derivation of $P(X=r) = \frac{e^{-\lambda}\lambda^{r}}{r!}$ $\quad r = 0, 1, 2 \ldots n.$

De-Moivere Laplace theorem : Suppose n is very large, p held fixed. $nC_r p^r q^{n-r} \sim \frac{1}{\sqrt{2\pi npq}} e^{-(r-np)^2/2npq}$.

$\mu = np$

$\sigma = \sqrt{npq}$

$\sigma^2 = npq$.

$\frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)}$.

where $p+q=1$

$$P(r_1 \leq X \leq r_2) = \int_{r_1}^{r_2} \frac{1}{\sqrt{2\pi\, npq}} e^{-(x-np)^2/2npq}\, dx$$

$$= \int_{t_1}^{t_2} \phi(t)\, dt.$$

$$t_1 = \frac{r_1 - np}{\sqrt{npq}} \qquad t_2 = \frac{r_2 - np}{\sqrt{npq}}$$

$$\phi(t) = \frac{1}{2\pi} e^{-t^2/2}$$

Probability of r successes in 'n' trials is reduced to evaluation of the normal curve which is given by

$$\phi(t) = \frac{1}{\sqrt{2\pi npq}} e^{-(x-np)^2/2npq}.$$

Q) A fair coin is tossed 1000 times. Find the probability that heads will show 510 times.

Sol. $p = \frac{1}{2}$ $\quad q = \frac{1}{2}$

Mean $= np$

$= 1000 \times \frac{1}{2}$

$= 500$

Variance $= npq = 500 \times \frac{1}{2}$

$= 250$

$r = 510$

$$P_{510}(H) = P(X=r) = \frac{1}{\sqrt{2\pi\, npq}} e^{-(x-np)^2/2npq}$$

$$= \frac{1}{\sqrt{2\pi(250)}} e^{-(510-500)^2/2\times 250}$$

$$= \frac{1}{\sqrt{2\times 3.14 \times 250}} e^{-(510-500)^2/500}$$

$$= 0.0207.$$

Conditional Distribution : Probability of event A assuming M is given by $P(A/M) = \frac{P(A \cap M)}{P(M)}$ where $P(M) \neq 0$

Let $F(X/M)$ denote conditional distribution of a random variable X assuming M is defined as the conditional probability of the event $\{X \leq x\}$

$$P(X/M) = P\{X \leq x/M\} = \frac{P\{(X \leq x) \cap M\}}{P(M)}$$

Properties of conditional Distribution $(F(X/M))$:

1) $F(\infty/M) = 1$
2) $F(-\infty/M) = 0$
3) $P\{(x_1 \leq X \leq x_2)/M\} = \frac{P\{(x_1 \leq X \leq x_2) \cap M\}}{P(M)}$

$$= F(x_2/M) - F(x_2/M)$$

Conditional Density :

Density function is the derivative of distribution function

$$\frac{d}{dx} F(X/M) = f(X/M)$$

$$f(X/M) = \lim_{\Delta x \to 0} \{x \leq X \leq x + \Delta x/M\}$$

Function of One Random Variable: Let X be a random variable with associated sample space S_X and a known probability distribution.

- The expression $Y = g(X)$ defines a new random variable X. $g(x)$ is a functional of real variable x.

$$S_Y = \{y = g(x) ; x \in S_X\}$$

- For $g(x)$ to be a random variable the function $g(x)$ must have the following property.

(i) Its domain must include the range of random variable X

(ii) The events $\{g(x) = \pm\infty\}$ must have zero probability.

To find PDF of Y if PDX of X is given.

Let X be a continuous random variable with pdf $f_X(x)$

$$Y = g(x) \text{ — (1)} \quad \text{pdf } f_Y(y)$$

$g(x)$ is a strictly monotic function of x,

$$x = g^{-1}(y)$$

diff (1) w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} g(x)$$

$$\frac{dy}{dx} = g'(x)$$

case 1: If $g(x)$ is strictly increasing function of X

$$f_y(y) = \frac{dx}{dy} f_x(x) \quad \text{--- (1)}$$

case 2: If $g(x)$ is strictly decreasing function of x

$$f_y(y) = -\left(\frac{dx}{dy}\right) f_x(x) \quad \text{--- (2)}$$

combining (1) & (2)

$$f_y(y) = f_x(x)\left|\frac{dx}{dy}\right|$$

Q) Let X be a continuous random variable with pdf ~~f(x)=~~

$$f(x) = \begin{cases} x/12 \; ; & 1<x<5 \\ 0 \; ; & \text{elsewhere} \end{cases}$$ Find the pdf of $Y = 2X-3$.

sol: $x = \frac{y+3}{2}$

$$\frac{dx}{dy} = \frac{1}{2}$$

$$f_x(x) = \frac{x}{12}$$

$$f_y(y) = f_x(x)\left(\frac{dx}{dy}\right)$$

$$= \frac{x}{12} \times \frac{1}{2}$$

$$= \frac{x}{24}$$

$$f_y(y) = \left(\frac{y+3}{48}\right)$$

$y = 2x-3$

$y = 2(1)-3$

$y = -1$

$y = 2(5)-3$

$y = 7$

$$f_y(y) = \begin{cases} \frac{y+3}{48} \; ; & -1<y<9 \\ 0 \; ; & \text{elsewhere} \end{cases}$$

Q) Given the random variable X with pdf $f_x = \begin{cases} 2x \; ; & 0<x<1 \\ 0 \; ; & \text{elsewhere} \end{cases}$

Find pdf of $y = 8x^3$.

sol: $x^3 = \frac{y}{8}$

$$x = \left(\frac{y}{8}\right)^{1/3}$$

$$x = \frac{y^{1/3}}{2}$$